Linear Programming in Practice

- **Essential Issue:** To model non-linear reality with linear equations
  - Activities
  - Piece-wise linear approximations
  - Fixed charges

- **An issue that may come up:** Duality

- **Some Example applications**
Motivation:

If we use a standard production function

\[ f(X) = \sum c_i X_i = Z \]

resources \( \Rightarrow \) output

We are not able to represent typical production function with diminishing marginal returns and non-linear isoquants

Concept

An activity is a specific way to use resources in fixed proportions

Physical interpretation is direct, e.g.:

- an aircraft using pilots, fuel / ton-km
- a machine requiring labor, materials per unit product

Think of activities as intermediates between resources and output

resources \( \Rightarrow \) activities \( \Rightarrow \) output
Example for 1 Activity

transport process $A_1$ uses 40 persons, 200 gallons to produce 100 Ton-km

Two Activities

$A_1 = (40p, 200g) \implies 100 \text{ T-m}$

$A_2 = (10p, 200g) \implies 50 \text{ T-m}$

$A_1 = \frac{1}{2} \quad (20, 100) \implies 50$

$A_2 = 1 \quad (10, 200) \implies 50$

$[A_1, A_2] = \left[ \frac{1}{2}, 1 \right]$

$\quad (30, 300) \implies 100$

Note: Isoquant horizontal and vertical above, below activities -- Why?
Many Activities

LP Formulation with Activities (1)

- Example: Maximize Profits from Production of Alloys,
  - 3 possible processes
  - limited by resources on hand (Crome and Carbon)
  - Different profitability for each process

- Optimize: Profit = \sum c_i P_i -- subject to constraints

<table>
<thead>
<tr>
<th>Element</th>
<th>Process 1</th>
<th>Process 2</th>
<th>Process 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cr</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Profit</td>
<td>30</td>
<td>28</td>
<td>29</td>
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</tbody>
</table>
LP Formulation with Activities (2)

\[ \text{max } Z = 30P_1 + 28P_2 + 29P_3 \]

\[ \text{s.t. } \begin{align*} 
6P_1 + 5P_2 + 3P_3 & \leq 26 \quad \text{(Cr)} \\
4P_1 + 2P_2 + 6P_3 & \leq 7 \quad \text{(C)} 
\end{align*} \]

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Piece-Wise Linear Approximations (1)

- **Motivation:**
  - Returns to scale generally non-linear
  - Straight line approximations are inaccurate
Piece-Wise Linear Approximations (2)

- Concept:
  - Represent \( f(X_1) \) with several lines

\[
\begin{align*}
&c_1 X_1 \\
&c_{1A} X_1 \\
&c_{1B} X_1 \\
&c_{1C} X_1
\end{align*}
\]

Piece-Wise Linear Approximations (3)

Implementation Notes:

- \( X_1 \) must be redefined as several variables: \( X_{1A}, X_{1B}, \ldots \)
- These new variables must not overlap, so \( X_{1A} < X_{1B}, \) etc.
- New variables and constraints make the LP larger and, thus more expensive
Piece-wise Linear Approximations (4)

- **Given:** Max $Z = f(X_1) + 4X_2$
  s.t. $3X_1 + 6X_2 \leq 8$

- **Piece-wise linear approximation gives:**
  - $X_1 \Rightarrow X_{1A} + X_{1B}$
  - $X_{1A}, X_{1B}$ have same $a_{ij}$ as $X_1$
  - $c_1 = c_{1A}, c_2A$
  - $X_{1A} < \text{cutoff X value between } X_{1A} \text{ and } X_{1B}, X'$

- **Thus:** Max $Z = c_{1A} X_{1A} + c_{1B} X_{1B} + 4X_2$
  s.t. $3X_{1A} + 3X_{1B} + 6X_2 \leq 8$
  $X_{1A} \leq X'$

Piece-wise Linear Approximations (5)

- **Key Limitation:**
  - ONLY works for convex feasible region!

- **Why?**
  - What if $c_{1B} > c_{1A}$? (see fig)

- **Max** $Z = c_{1A} X_{1A} + c_{1B} X_{1B} + 4X_2$

- **The LP will select** $X_{1B}$ before $X_{1A}$

Result may be meaningless!

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Engineering Systems Analysis for Design
Richard de Neufville, Joel Clark, Frank R. Field
Massachusetts Institute of Technology
LP in Practice
Slide 13 of 25
Convex Feasible Regions Review:

Piecewise linear approximation works when FR is convex

- Convex Feasible Region
- Non-Convex Feasible Region

Fixed Charges

- Example: Warehousing
  - Cost = fixed rent, etc. + variable
  - Unless you choose not to operate it!

  \[
  f(X_1) = c_0 + c_1 X_1 \quad X_1 \geq 0
  \]
  \[
  f(X_1) = 0 \quad X_1 = 0
  \]

  \[
  \text{LP generally cannot handle fixed charges}
  \]

  
  Exception:
  - All \( X_i > 0; \ X_i \neq 0 \)
  - then subtract \( \Sigma c_0 \)
  - and optimize
Duality

- Concept:
  - A “dual” is a mirror-image form to another problem (the “primal”)
  - If primal = max; then dual = min
  - If primal = min; then dual = max
  - Dual contains all information of the primal, but in a different format
  - Optimum value of primal = optimum value of dual

- Example:
  - Primal: maximize output subject to budget limitations
  - Dual: minimize costs subject to output requirements

LP Duality

- Mathematics:
  - Given a Primal:
    - Optimize: \( Z = c^T X \)
    - subject to: \( A X \leq \geq B \)
  - Dual is:
    - Optimize: \( Y = B^T W \)
    - subject to: \( A^T W \leq \geq c^T \)

- Change of dimensionality between primal & dual:
  - \( c^T \) and \( B \) have different number of variables

- Can use duality to:
  - Reduce size of constraint matrix
  - Speed up LP solution
LP Duality - Example (1)

Primal: Max: \( Z = X_1 + 2X_2 + 3X_3 \)
\[ \text{s.t.} \quad \begin{align*}
4X_1 + 2X_2 & \leq 5 \\
6X_1 + 7X_2 + 9X_3 & \leq 12
\end{align*} \]

\[ A = \begin{bmatrix} 4 & 2 & 0 \\ 6 & 7 & 9 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 6 \\ 2 & 7 \\ 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 12 \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

So: Max: \( Z = CX \)
\[ \text{s.t.} \quad AX \leq B \]

LP Duality - Example (2)

Primal: Max: \( Z = X_1 + 2X_2 + 3X_3 \)
\[ \text{s.t.} \quad \begin{align*}
4X_1 + 2X_2 & \leq 5 \\
6X_1 + 7X_2 + 9X_3 & \leq 12
\end{align*} \]

Dual: Min: \( Y = 5W_1 + 12W_2 \)
\[ \text{s.t.} \quad \begin{align*}
4W_1 + 6W_2 & \geq 1 \\
2W_1 + 7W_2 & \geq 2 \\
9W_2 & \geq 3
\end{align*} \]

So: Max: \( Z = CX \)
\[ \text{s.t.} \quad AX \leq B \]
Min: \( Y = B^T W \)
\[ \text{s.t.} \quad A^T \leq C^T \]
LP Duality - Interpretation of Results

--Primal:
Max: \( Z = 3X_1 + X_2 + 8X_3 \)
\[ \text{s.t. } \begin{align*}
X_1 + X_2 + X_3 &\leq 4 \\
X_1 + X_2 + X_3 &\leq 7 \\
2X_2 + X_3 &\leq 8
\end{align*} \]
\[ X^* = \{0,2,4\} \]
\[ \text{SP}^* = \{7.5,0,0.5\} \]
\[ \text{OC}^* = \{4.5,0,0\} \]
\[ \text{SV}^* = \{0,1,0\} \]
\[ Z^* = 34 \]

--Dual:
Min: \( Y = 4W_1 + 7W_2 + 8W_3 \)
\[ \text{s.t. } \begin{align*}
W_1 + W_2 &\geq 3 \\
W_2 + 2W_3 &\geq 1 \\
W_1 + W_2 + W_3 &\geq 8
\end{align*} \]
\[ W^* = \{7.5,0,0.5\} \]
\[ \text{dSV}^* = \{4.5,0,0\} \]
\[ \text{dSP}^* = \{0,2,4\} \]
\[ \text{dOC}^* = \{0,1,0\} \]
\[ Y^* = 34 \]

Primal-Dual Relationships in Solution

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision Variables</td>
<td>Shadow Prices</td>
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<td>Decision Variables</td>
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<tr>
<td>Opportunity Costs</td>
<td>Slack Variables</td>
</tr>
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<td>Opportunity Costs</td>
</tr>
</tbody>
</table>
Some Real-World Applications (1)

- **Airline Scheduling**
  - **Objective:** Minimize Cost
  - **Constraints:**
    - Number of Aircraft
    - Available time on Aircraft before Maintenance
    - Crews Available
    - Limits on Crew time on duty
    - Location of crews and aircraft
    - Traffic between points, etc, etc


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Some Real-World Applications (2)

- **Production and Logistics**
  - **Objective:** Minimize Cost / Maximize Throughput
  - **Constraints:**
    - Number and Capacity of Facilities
    - Connectivity of Network
    - Personnel restrictions
    - Location of crews and aircraft
    - Times orders are made
    - Delays in system, etc, etc,

Summary on Applications

• Special Formulations required so that LP can represent realistic problems
  • Activity representations
  • Piece-wise linear approximations
  • Integers (not discussed here)
• Much sophistication in mathematics possible
  • Duality gives a flavor
  • See Example applications
• However, LP basically deals with system models with known parameters, without risk

MORE WILL BE REQUIRED!!!