Linear Programming
Sensitivity Analysis

Sensitivity Analysis

- Rationale
- Shadow Prices
  - Definition
  - Use
  - Sign
  - Range of Validity
- Opportunity Costs
  - Definition
  - Use
Rationale for Sensitivity Analysis

- Math problem is an approximation
  - optimum is an approximation
  - we need to check

- Constraints often artificial
  - Designer should question them
  - Should we have different specifications?

- Situations always probabilistic
  - Prices change
  - Need to assess risk

Shadow Price Definition

- Recall from Constrained Optimization:
  - Shadow price = $\frac{\delta (\text{objective function})}{\delta (\text{constraint})}$ at the optimum

- Complementary Slackness:
  Either (Slack variable) or (shadow price) = 0
Shadow Price Illustration

Max: \( Y = X_1 + 4X_2 \)

s.t.
\[
\begin{align*}
X_1 + X_2 & \leq 5 = b_1 \\
X_1 & \geq 3 = b_2 \\
X_2 & \leq 3 = b_3 \\
X_1, X_2 & > 0
\end{align*}
\]

Notes:
\begin{enumerate}
\item \( X_1^* = 3; X_2^* = 2; Y^* = 11 \)
\item when \( \Delta b_1 = \pm 1 \)
\( \Delta X_2^* = \pm 1; \Delta Y^* = \pm 4; SP_1 = 4 \)
\item SP_3^* = 0; slack_3 = 1
\item when \( b_1 > 6 \)
\( \text{slack}_3 = 0; SP_3 \neq 0; SP_1 = 1 \leq 4 \)
\end{enumerate}

Proactive Use of Shadow Prices

- Identify constraints with high S.P

- See if they can be changed for better solutions

- Example: New York water supply
  - Original Design for Third City Tunnel ($1 billion plus)
  - pressure < 40 psi at curb (some point in Brooklyn)
  - No allowance for local tanks, pumps
  - Shadow price in millions of dollars!
Reactive Use of Shadow Prices

- Respond to new opportunities
- Example: client changes specifications

- Respond to proposals for new constraints
- Example: trace chemicals

Sign of Shadow Prices

- "Obvious Rule" (+SP with +Δb) not correct

- Correct Reasoning:
  - What makes the optimum better?
    - Expansion of feasible region => "Relaxation of constraints"
  - What changes will increase the feasible region?
    - Increase upper bound: \( \sum_j a_{ij}X_j < b_i \)
    - Decrease lower bound: \( \sum_k a_{kj}X_j > b_k \)

  - i.e., "Raise the roof, lower the floor."
Shadow Price Illustration

Max: \( Y = X_1 + 4X_2 \)

s.t. \( X_1 + X_2 \leq 5 = b_1 \)
\( X_1 \geq 3 = b_2 \)
\( X_2 \leq 3 = b_3 \)
\( X_1, X_2 \geq 0 \)

Notes:

a) \( X_1^* = 3; X_2^* = 2; Y^* = 11 \)

b) when \( \Delta b_1 = \pm 1 \)
\( \Delta x_1^* = \pm 1; \Delta y^* = \pm 4; \) \( SP_1 = 4 \)

c) \( SP_3^* = 0; \) slack_3 = 1

d) when \( b_1 > 6 \)
\( \) slack_3 = 0; \( SP_3 \neq 0; \) \( SP_1 = 1 \leq 4 \)

Shadow Prices As Constraints Change

increase an upper bound ("raise the roof")

decrease a lower bound ("lower the floor")

When \( b_2: 3 \rightarrow 2 \)
new \( X^* = [2,3] \)
new \( Y^* = 14 \)
\( \Delta Y^* = 3 \)
Range of Shadow Prices

- In Linear Programming, Shadow prices are constant.
- Until a constraint changes enough so that a new constraint is binding.
- Results given as:

  \[ SP_K = \text{constant} \]

  for \( r_L < b_K < r_U \)

- Outside the range:
  - Shadow prices decrease as constraint is relaxed.
  - Shadow prices increase as constraint is tightened.

Shadow Price Ranges for Example

Max: \( Y = X_1 + 4X_2 \)

s.t. \( X_1 + X_2 \leq 5 = b_1 \)
    \( X_1 \geq 3 = b_2 \)
    \( X_2 \leq 3 = b_3 \)
    \( X_1, X_2 \geq 0 \)

Shadow Prices

\( SP_1 = 4 \quad 3 \leq b_1 \leq 6 \)
\( SP_2 = 4 \quad 2 \leq b_2 \leq 5 \)
\( SP_3 = 0 \quad 2 \leq b_3 \)
Opportunity Cost - Definition

- Objective Function $= \sum c_i X_i$
- Opportunity costs associated with $c_i$ -- the coefficients of design/decision variables
- At optimum, some decision variables $= 0$
  - These are non-optimal decision variables
- Opportunity cost is:
  - Degradation of optimum per unit of non-optimal variable introduced into design
  - A "cost" in that it is a *worsening* of optimum. Units may be almost anything; equal to whatever units are being optimized.

Meaning of Opportunity Costs

- Opportunity cost defines design trigger "price"
  - The value of the coefficient of the decision variable for which that variable should be in the design
- Suppose: Obj.Function $= \ldots + c_K X_K + \ldots$
  and $X_K$ not optimal with an opportunity cost $= OC_K$
- Then, as $c_K$ changes for the better, (greater for maximization, lesser for minimization)
  - $OC_K$ lower
  - $OC_K = 0$ at $c'_K = c_K - OC_K$
- $c'_K$ is trigger price; defines the limit of best design
Illustration of Opportunity Cost

- What happens when forced to use a non-optimal decision variable?
- Example: Min Cost = 2X₁ + 10X₂ + 20X₃
  \[\begin{align*}
  \text{s.t.} & \quad X₁ + X₂ + X₃ \geq 3 \\
  & \quad X₂ \geq 1 \\
  & \quad X₁, X₂, X₃ \geq 0
  \end{align*}\]
  - \(X^* = (2, 1, 0);\) cost* = 14
  - If forced to use \(X₃\), new \(X^* = (1, 1, 1);\) new cost* = 32
  
  Thus: (opportunity cost)₃ = \(\Delta Z^*/1 = 18\)

Use of Opportunity Cost

- At what price would it be desirable to use \(X₃\) ?
  - If \(X₃\) is used with no change in its unit cost (= \(c₃\)), the optimal cost would increase by 18
  - If the cost of \(X₃\) were to fall by an amount equal to the opportunity cost (\(c₃' = c₃ - OC₃ = 20 - 18 = 2\)). It would then compete with \(X₁\)
  - So the answer is: When its unit cost falls by its opportunity cost: 20 - 18 = 2
How do you find SP and OC?

- LP optimization programs all calculate shadow prices and opportunity costs routinely and “print them out” for you

- Sometimes, programs report this information in special ways. Thus:
  - Shadow Prices \( \leq \) "dual decision variables"
  - Opportunity Costs \( \leq \) "dual slack variables"
  - More on this later

A Possible Semantic Confusion

- Note that the Phrases “shadow price” and “opportunity cost” have somewhat different meanings in LP and Economics literature

- The “opportunity cost” of an action in economics can be interpreted as the “shadow price” of that action on the budget...
Summary on LP Sensitivity Analysis

• LP Optimization Programs automatically provide important information useful for improving/changing design
• Shadow prices -- to help redefine constraints
• Opportunity costs -- to identify critical prices

• Need to understand these quantities carefully