Constrained Optimization

- Unconstrained Optimization (Review)
- Constrained Optimization
  - Approach
  - Equality constraints
    - Lagrangeans
    - Shadow prices
  - Inequality constraints
    - Kuhn-Tucker conditions
    - Complementary slackness

Unconstrained Optimization (1)

- Definitions:
  - Optimization = Maximum of desired quantity
    = Minimum of undesired quantity
  - Objective Function = Expression to be optimized
    = \( Z(\mathbf{X}) \)
  - Decision Variables = Variables about which we can make decisions
    = \( \mathbf{X} = (X_1, \ldots, X_n) \)
Unconstrained Optimization (2)

- By calculus:

If \( F(X) \) continuous, analytic:
Condition for maxima and minima
\[ \frac{\partial F(X)}{\partial X_i} = 0 \quad \forall_i \]

Unconstrained Optimization (3)

- Secondary conditions:

\[ \frac{\partial^2 F(X)}{\partial X_i^2} < 0 \quad \Rightarrow \text{Max} \quad (B,D) \]
\[ \frac{\partial^2 F(X)}{\partial X_i^2} > 0 \quad \Rightarrow \text{Min} \quad (A,C,E) \]

These define whether point of no change in \( Z \) is a maximum or a minimum
Unconstrained Optimization (4)

- Example: Housing insulation
  \( F(x) = \frac{K_1}{x} + K_2x \)
  Total Cost = Fuel cost + Insulation cost
  \( x = \) Thickness of insulation

  \[
  \frac{\partial F(x)}{\partial x} = 0 = -\frac{K_1}{x^2} + K_2
  \]

  \[ x^* = \left\{ \frac{K_1}{K_2} \right\}^{1/2} \]
  (starred quantities are optimal)

Unconstrained Optimization (5)

- \( K_1 = 500 \quad K_2 = 24 \quad X^* = 4.56 \)
**Constrained Optimization**

- "Constrained Optimization" involves the optimization of a process subject to constraints.
- Constraints have two basic types:
  - Equality Constraints -- some factors have to equal constraints.
  - Inequality Constraints -- some factors have to be less than or greater than the constraints (these are "upper" and "lower" bounds).

**Equality Constraints**

- Example: Best use of budget
- Maximize: Output $= Z(X) = a_0 x_1^{a_1} x_2^{a_2}$
- Subject to (s.t.):
  
  $$\text{Total costs} = \text{Budget} = p_1 x_1 + p_2 x_2$$

Note: $\frac{\partial Z(X)}{\partial X} \neq 0$ at optimum
Constrained Optimization

- Approach
  To solve situations of increasing complexity, (for example, those with equality, inequality constraints) ...

  Transform more difficult situation into one we know how to deal with

- In this case, transform optimization of a “constrained” situation to optimization of “unconstrained” situation

Lagrangean Method (1)

- Transforms equality constraints into unconstrained problem

- Start with:
  \[
  \begin{align*}
  \text{Opt: } & F(x) \\
  \text{s.t.: } & g_i(x) = b_j \Rightarrow g_i(x) - b_j = 0
  \end{align*}
  \]

- Get to:
  \[
  L = F(x) - \sum \lambda_j [g_j(x) - b_j]
  \]

  \(\lambda_j\) = Lagrangean multipliers (lambdas) -- these are unknown quantities for which we must solve

  Note: \([g_j(x) - b_j] = 0\) by definition, thus
  optimum for \(F(x) = \text{optimum for } L\)
Lagrangean Method (2)

To optimize $L$:

\[
\frac{\partial}{\partial x_i} L = 0 \quad \forall_i \\
\frac{\partial}{\partial \lambda_j} L = 0 \quad \forall_i
\]

Example:

Opt: $F(x) = 6x_1x_2$

s.t.: $g(x) = 3x_1 + 4x_2 = 18$

$L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18)$

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 6x_2 - 3\lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 6x_1 - 4\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 3x_1 + 4x_2 - 18 = 0
\end{align*}
\]

Solving as unconstrained problem:

\[
\begin{align*}
\lambda &= 2x_2 = 1.5x_1 \\
x_2 &= 0.75x_1 \\
3x_1 + 3x_1 - 18 &= 0
\end{align*}
\]

\[
\begin{align*}
x_1^* &= 18/3 = 6 \\
x_2^* &= 18/8 = 2.25 \\
\lambda^* &= 4.5 \\
F(x)^* &= 40.5
\end{align*}
\]
Shadow Prices

- Shadow Price is the Rate of change of objective function per unit change of constraint
  \[ \frac{\partial F(x)}{\partial b_j} \]
- This is meaning of Lagrangean multiplier
  \[ SP_j = \frac{\partial F(x)^*}{\partial b_j} = \lambda_j \]
  Naturally, this is an instantaneous rate
- The shadow price is extremely important for system design
- It defines value of changing constraints

Shadow Prices (2)

- Let’s see how this works in example, by changing constraint by 0.1 units:
  Opt: \[ F(x) = 6x_1x_2 \]
  s.t.: \[ g(x) = 3x_1 + 4x_2 = 18.1 \]
- The optimum values of the variables are
  \[ x_1^* = (18.1)/6 \quad x_2^* = (18.1)/8 \]
- Thus \[ F(x)^* = 6(18.1/6)(18.1/8) = 40.95 \]
  \[ \Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1) \]
Inequality Constraints

- Example: Housing insulation
  Min: Costs = $K_1 / x + K_2 x$
  s.t.: $x \geq 8$ (minimum thickness)

![Optimizing Cost Example]

Inequality Constraints (2)

- Approach: Transform inequalities into equalities, then proceed as before

- Again, introduce new variable -- the “Slack” variable that defines “slack” or distance between constraint and amount used

- The resulting equations are known as the “Kuhn-Tucker conditions”
Inequality Constraints -- insertion of slack variables in Lagrangean

- A “slack variable”, $s_j$, for each inequality
  
  $g_j(x) \leq b_j \Rightarrow g_j(x) + s_j^2 = b_j$
  $g_j(x) \geq b_j \Rightarrow g_j(x) - s_j^2 = b_j$

- These are “squared” to be positive

- start from:
  
  opt: $F(x)$ s.t.: $g_j(x) \leq b_j$

- get to:
  
  $L = F(x) - \sum \lambda_j [g_j(x) + s_j^2 - b_j]$

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Inequality Constraints -- Complementary Slackness Conditions

- The optimality conditions are:
  
  $\frac{\partial L}{\partial x_i} = 0$
  $\frac{\partial L}{\partial \lambda_j} = 0$
  plus: $\frac{\partial L}{\partial s_j} = 2s_j \lambda_j = 0$

- These new equations imply:
  
  $s_j = 0 \quad \lambda_j \neq 0$
  or
  $s_j \neq 0 \quad \lambda_j = 0$

  They are the “complementary slackness” conditions. Either slack or lambda =0 ∀i
Interpretation of Complementary Slackness Conditions

- Interpretation:
  - If there is slack on $b_j$, (i.e. more than enough of it)
    => No value to objective function to having more: $\lambda_j = \frac{\partial F(x)}{\partial b_j} = 0$

- If $\lambda_j \neq 0$, then all available $b_j$ used
  => $s_j = 0$

Application to Example

- Min: Costs = $K_1 / x + K_2 x$
  s.t.: $x \geq 8$ (minimum thickness)

$L = K_1 / x + K_2 x - \lambda[x - s^2 - b]$
$L = 500 / x + 24x - \lambda[x - s^2 - 8]$
$500 / x^2 + 24 - \lambda = 0 \quad 2 \lambda s = 0 \quad x - s^2 = 8$
- If $s = 0$, $x = 8$, $\lambda = 31.8$ (at that point)
  Max = 254.5
- Therefore, worth relaxing (in this case, lowering) constraint to get maximum
- $x^* = 4.56 \quad$ Optimum = 221
Unconstrained Optimization (5)

- \( K_1 = 500 \quad K_2 = 24 \quad X^* = 4.56 \)

Another application to Example

- Min: Costs = \( K_1 / x + K_2x \)
  s.t.: \( x \geq 4 \) (NEW MINIMUM)
- \( L = K_1/x + K_2x - \lambda(x + s^2 - b) \)
- \( L = 500/x + 24x - \lambda(x + s^2 - 4) \)
- \( 500/x^2 + 24 - \lambda = 0 \quad 2\lambda s = 0 \quad x - s^2 = 4 \)
- If \( \lambda = 0, \quad x = 4.56, \quad \text{slack, } s^2 = 1.56 \)
  Optimum = 221
- not worth changing constraint
Summary of Presentation

- Important mathematical approaches
  - Lagrangeans
  - Kuhn-Tucker Conditions

- Important Concept: Shadow Prices

- THESE ANALYSES GUIDE DESIGNERS TO CHALLENGE CONSTRAINTS