Real Options Course

Pricing Options by Simulation

Today’s plan

- The flaw of the average
- Monte Carlo Simulation
  - Single numbers ain’t good enough
- Modelling stock prices
  - valuing European call options
- Modelling decisions
  - valuing a growth option
The Flaw of the Average

- Plugging average values into uncertain cells can lead you astray
  - Let us look at an example
- The resulting bottom line (e.g. NPV) is often not the average
- Mathematical Reason: $E(f(X))=f(E(X))$ for a random variable $X$ holds ONLY if $f$ is linear

A few words about modelling…

- Why spreadsheets?
- Spreadsheets have many disadvantages
  - Limited data structure (2-dimensional array)
  - Difficult to validate and document
  - Inflexible
  - Partially unreliable numerical routines
The Cons

And the Pros

30 MILLION USERS
The five stages of computer modelling
(Donald Knuth)

1. Decide what you want the model to do
2. Decide how to build the model
3. Build the model
4. Debug the model
5. Trash stages 1 to 4 and start again, now that you know what you really wanted in the first place

Don’t get frustrated: A modelling process is a learning process

The main benefit of building a (computer) model to analyse a problem is not the quantitative information obtained as output of the model but the enhanced understanding of the problem gained during the modelling process
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What do we want to achieve with a simulation model?

- Option pricing: what is the value of an option?
- But: The value of an option is a random variable
  - A single number, e.g. an average, gives very limited information about a random variable
- Manual what-if analysis is cumbersome and biased
- We want to estimate the distribution of NPV
  - Give a graphical representation of this distribution
  - cumulative distribution function
  - histogram
Example: Value at risk

10% VAR is roughly £500,000
5% VAR is roughly £800,000

Preparing a spreadsheet model for simulation

- Write a model as if all inputs (data) were certain
- Mark clearly all uncertain input cells (colour them)
- Feed the input cells with appropriate randomly chosen numbers
  - The F9 key (recalculation) now produces one scenario after the other
- Replicate your model using the data table command
  - Let’s see how that works
Feeding uncertain cells

- Suppose cell \( x \) is known to be uniformly distributed on the interval \([0,1]\)
  - Put \( "=\text{rand}()" \) into cell \( x \)
  - Pressing F9 is equivalent to sampling from a uniform distribution and putting the number into cell \( x \)
- \( \text{randbetween}(a,b) \) samples integers between \( a \) and \( b \), including the integers \( a \) and \( b \), with equal probability \( 1/(b-a+1) \)
- Analysis Tool Pack needs to be loaded for this to work (Tools \( \rightarrow \) add-ins)

More general distributions

- Inverse function method can be used to generate more general distributions using the \( \text{rand}() \) function
  - Example: \( \text{norminv}(\text{rand}(), a, b) \) samples from a normal distribution with mean \( a \) and standard deviation \( b \)
- Cheap alternatives: random variable generators are provided as Excel add-ins with most modern OR books
- More professional alternatives: @Risk, Crystal Ball
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Example: Modelling Stock price movement

- Model:
  \[ S_{t+1} = u_t S_t, \quad t=0,1,2,\ldots \]

  where \( u_t \) are random variables
  - Assume \( u_t, \quad t=0,1,2,\ldots \) to be independent
  - Notice that \( u_t = S_{t+1}/S_t \) is independent of the units of measurement of stock price
  - Call \( u_t \) the return of the stock
- What is a realistic distribution for returns \( u_t \)?
An additive model

- Passing to logarithms gives
  \[ \ln S_{t+1} = \ln S_t + \ln u_t \]
- Let \( w_t = \ln u_t \)
- \( w_t \) is the sum of many small random changes between \( t \) and \( t+1 \)
- Central limit theorem: The sum of (many) random variables is (approximately) normally distributed (under typically satisfied technical conditions)
  - Most important result in probability theory
  - Explains the importance and prevalence of the normal distribution

Log-normal random variables

- Assume that \( \ln u_t \) is normal
  - Central limit theorem is theoretical argument for this assumption
  - Empirical evidence shows that this is a reasonably realistic assumption for stock prices
    however, real return distributions have often fatter tails
- If the distribution of \( \ln u \) is normal then \( u \) is called log-normal
  - Notice that log-normal variables \( u \) are positive since \( u = e^{\ln u} \) and with normally distributed \( \ln u \)
Distribution of return

- Assume that the distribution of $u_t$ is the same for all $t$
- Under log-normal assumption the distribution is defined by mean and standard deviation of the normal variable $\ln u_t$
  
  Growth rate $\nu=E(\ln u_t)$, Volatility $\sigma=\text{Std}(\ln u_t)$
- Typical values are
  \begin{align*}
  \nu &= 12\%, \quad \sigma = 15\% \text{ per annum} \\
  \nu &= 1\%, \quad \sigma = 1.25\% \text{ per month}
  \end{align*}
- Careful: if $\ln u$ is normal with mean $\nu$ and variance $\sigma^2$ then the mean of the log-normal variable $u$ is NOT $\exp(\nu)$ but $E(u) = \exp(\nu + \sigma^2/2)$ and $\text{Var}(u) = \exp(2\nu + \sigma^2)(\exp(\sigma^2)-1)$

Model of stock prices

\[ S_{t+1} = u_t S_t, \quad t=0,1,2,... \]
- $u_t$'s are independent log-normal random variables with
  
  \begin{align*}
  E(u) &= \exp(\nu + \sigma^2/2) \\
  \text{Var}(u) &= \exp(2\nu + \sigma^2)(\exp(\sigma^2)-1)
  \end{align*}
- Model is determined by growth rate $\nu$ and volatility $\sigma$, which are the mean and std of $\ln u_t$
- Values for $\nu$ and $\sigma^2$ can be found empirically by fitting a normal distribution to the logarithms of stock returns
Simulation

- Find \( \nu \) and \( \sigma \) for a basic time interval (e.g. \( \nu = 14\% \), \( \sigma = 30\% \) p.a.)
- Divide the basic time interval (e.g. a year) into \( m \) intervals of length \( \Delta t = 1/m \) (e.g. \( m = 52 \) weeks)
  - Time domain \( T = \{0, 1, \ldots, m\} \)
- Use model \( \ln S_{t+\Delta t} = \ln S_t + w_t \)
- By assumption
  - \( \ln S_m = \ln S_0 + w_1 + \ldots + w_m \)
  - \( w_1 + \ldots + w_m \) is \( N(\nu, \sigma^2) \)
- Assume all \( w_i \) are independent \( N(\nu',\sigma'^2) \) (CLT)
  - \( \nu = E(w_1 + \ldots + w_m) = mv', \text{ hence } \nu' = \nu/m \)
  - \( \sigma^2 = \text{Var}(w_1 + \ldots + w_m) = m \sigma'^2, \text{ hence } \sigma'^2 = \sigma^2/m \)
- Hence \( \ln S_{t+\Delta t} = \ln S_t + w_t \)
- \( w_t \) is normal with mean \( \nu \Delta t \) and variance \( \sigma^2 \Delta t \)

Simulation

- Model \( \ln S_{t+\Delta t} = \ln S_t + w_t \)
- \( w_t \) normal with mean \( \nu \Delta t \) and variance \( \sigma^2 \Delta t \)
- If \( Z \) is a standard normal variable (mean=0, var=1) then
  \[ \ln S_{t+\Delta t} = \ln S_t + \nu \Delta t + \sigma \sqrt{\Delta t} Z \]
- Equivalent: \( S_{t+\Delta t} = \exp(\nu \Delta t + \sigma \sqrt{\Delta t}) S_t \)
- Can use this equation to simulate process \( S_t \)
  - See spreadsheet model
Options

- European call option: “right but not obligation to buy share for £K at time T”
- Option value at time T is max(0,K-S)
- Price is the expected net present value of the option value
- Arbitrage argument (Black-Scholes, Merton):
  - Need to discount at the rate of growth of the stock (risk-adjusted discount rate)
- Equivalently: change growth rate of stock to risk-free rate and discount at risk-free rate
  - Risk neutral valuation
  - See spreadsheet model