Linear Programming in Practice

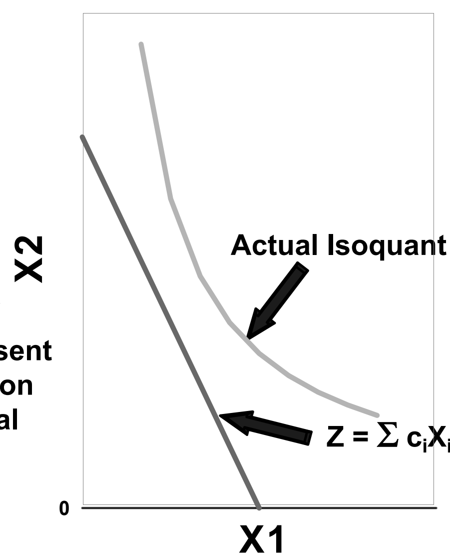
- Essential Issue: To model non-linear reality with linear equations
  - Activities
  - Piece-wise linear approximations
  - Fixed charges
- Another practical question
  - Duality

Activities

- Motivation:
- If we use a standard production function

  $f(\underline{X}) = \Sigma \ c_i X_i = Z$

  resources, ⟹ output
- We are not able to represent typical production function with diminishing marginal returns and non-linear isoquants

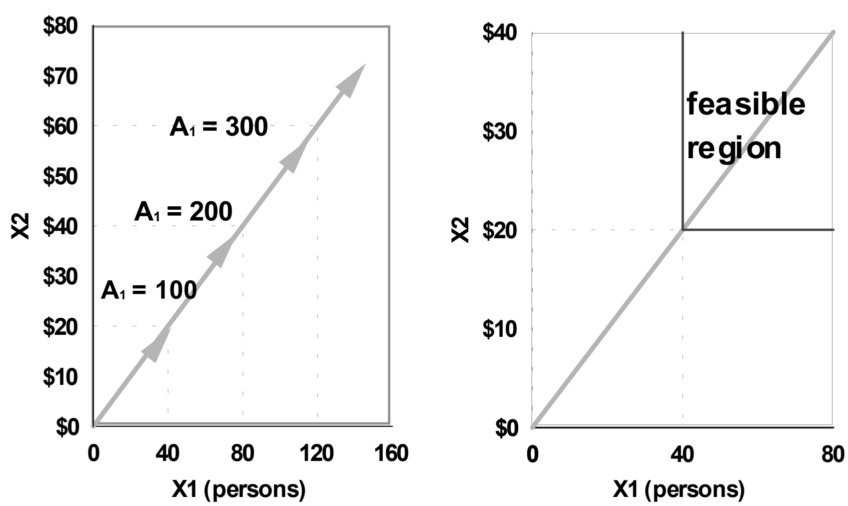


Activities (continued)

- **Concept**
  - **An activity is a**
    - *Specific way to use resources*
    - *in fixed proportions*
- **Physical interpretation is direct, e.g.:**
  - **an aircraft using pilots, fuel / ton-km**
  - **a machine requiring labor, materials per unit product**
- **Think of activities as intermediates between resources and output**

resources ⇒ activities ⇒ output

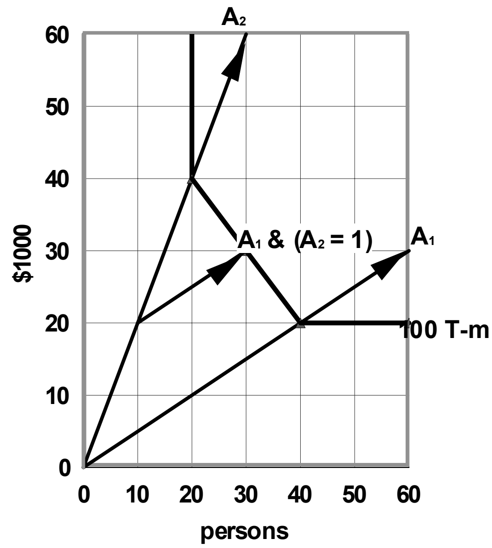
Example: transport process A_1 uses 40 persons, \$20k to produce 100 T-km



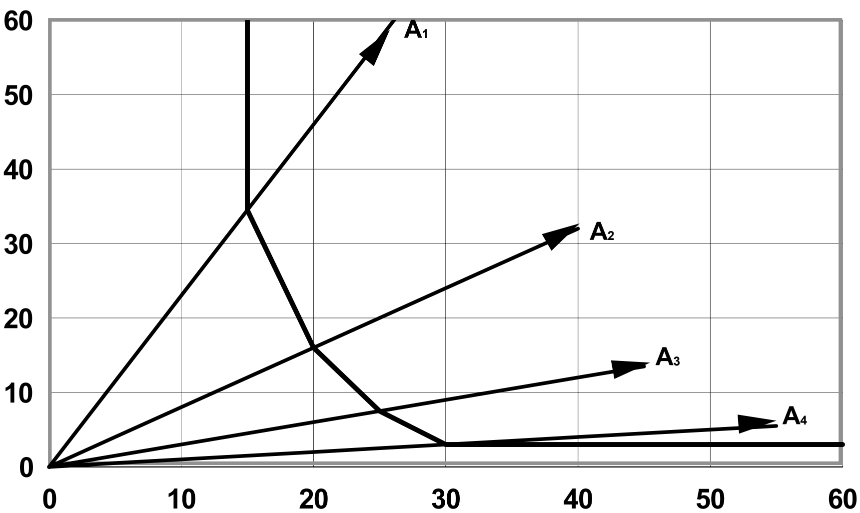
Activities

- Two Activities
- A_1 = (40,20K)
  ==> 100 T-m
- A_2 = (10,20K)
  ===> 50 T-m

A_2 = 1 {10, 20} => 50
A_1 = ½ {20, 10} => 50



Many Activities

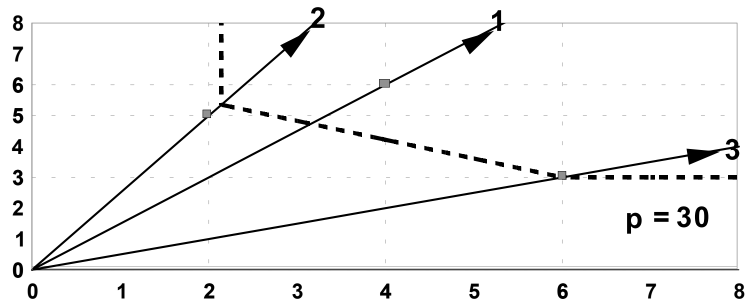


LP Formulation with Activities

	Cr (kg)	C (kg)	Profit ($)
Process 1	6	4	30
Process 2	5	2	28
Process 3	3	6	29

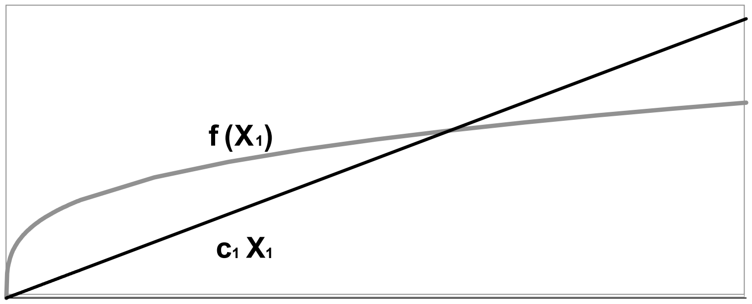
- Optimize: $Z = \Sigma c_i A_i$ -- subject to constraints
- Example: Alloy optimization
  - Three possible processes, each with different unit costs
  - subject to limits on Cr, C content

Formulation



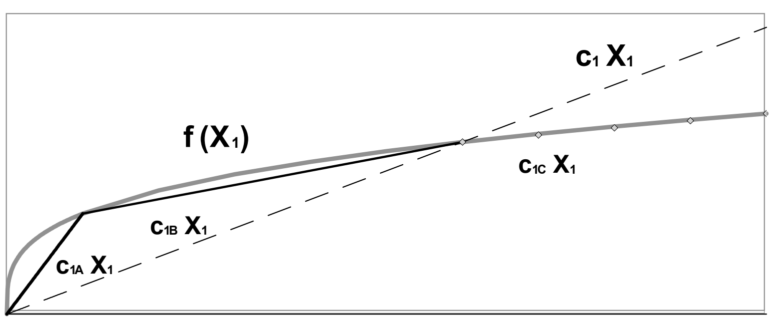
$$\max Z = 30 P_1 + 28 P_2 + 29 P_3$$
$$\text{s.t.} \quad 6 P_1 + 5 P_2 + 3 P_3 \leq 26 \text{ (Cr)}$$
$$4 P_1 + 2 P_2 + 6 P_3 \leq 7 \text{ (C)}$$

Piece-Wise Linear Approximations (1)



- **Motivation:**
  - –Returns to scale generally non-linear
  - –Straight line approximations are inaccurate

Piece-Wise Linear Approximations (2)

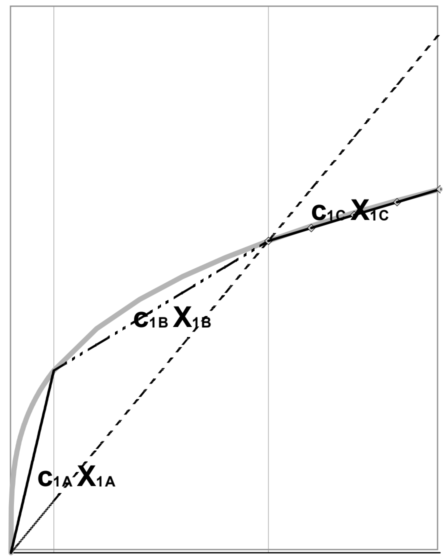


- **Concept:**
  - –Represent $f(X_1)$ with several lines

Piece-wise Linear Approximations

Implementation Notes:

- X1 must be redefined as several variables - X_{1A}, X_{1B}, ...
- These new variables must not overlap, so $X_{1A} < X_{1B}$, etc.
- New variables and constraints make the LP larger and, therefore, more expensive



Piece-wise Linear Approximations

- Mathematically:
- Given: Max $Z = f(X_1) + 4X_2$
  s.t. $3X_1 + 6X_2 \leq 8$
- Piece-wise linear approximation gives:
  - $X_1 \Rightarrow X_{1A} + X_{1B}$
  - X_{1A}, X_{1B} have same a_{ij} as X_1
  - $c_1 = c_{1A}$, c^{2A}
  - $X_{1A} <$ cutoff X value between X_{1A} and X_{1B}, X'
- Thus: Max $Z = c_{1A}\ X_{1A} + c_{1B}\ X_{1B} + 4X_2$
  s.t. $3\ X_{1A} + 3X_{1B} + 6X_2 \leq 8$
  $X_{1A} \leq X'$

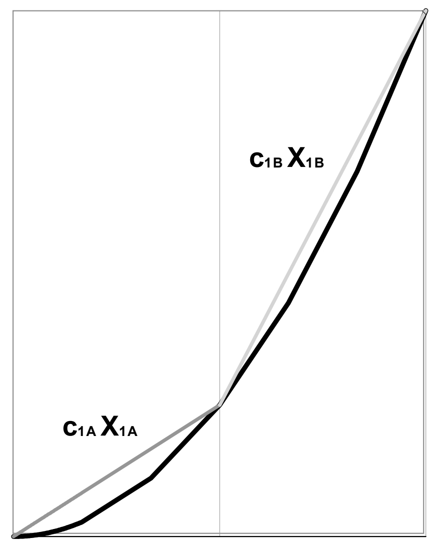
Piece-wise Linear Approximations (4)

- **Key Limitation:**
  - **–ONLY works for convex feasible region!**
- **Why?**
  - **–What if $c_{1B} > c_{1A}$? (see fig)**

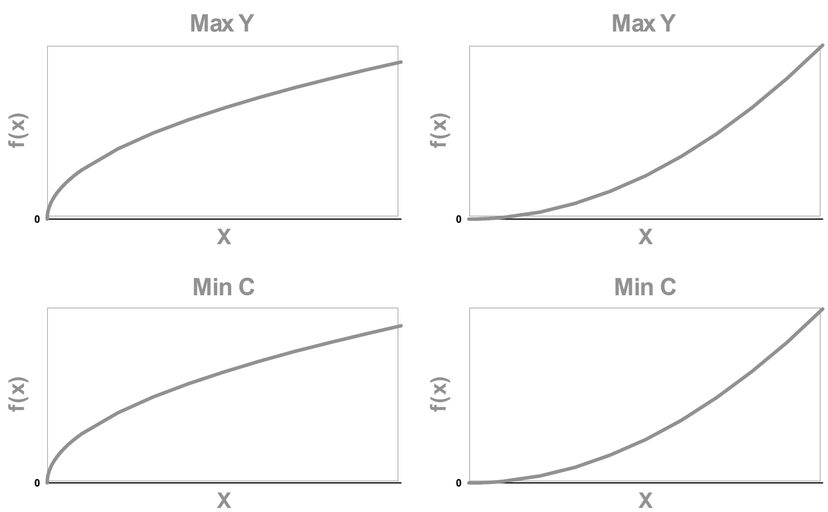
Max Z = $c_{1A}\ X_{1A} + c_{1B}\ X_{1B} + 4X_2$

- **The LP will select X_{1B} before X_{1B};**

Result may be meaningless!



Convex Feasible Regions Review:
Piecewise linear approximation works when FR is convex



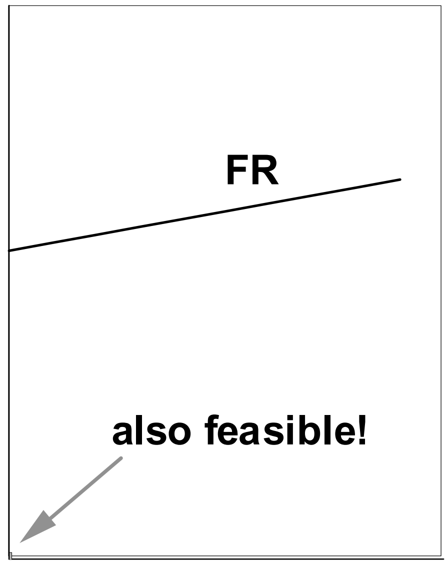
Fixed Charges

- Example: Warehousing
  - –Cost = fixed rent, etc. + variable
  - –<u>Unless</u> you choose not to operate it!

$f(X_1) = c_0 + c_1X_1 \qquad X_1 \geq 0$
$f(X_1) = 0 \qquad X_1 = 0$

- LP generally cannot handle fixed charges

  <u>Exception:</u>
  - –All $X_i > 0$; $X_i \neq 0$
  - –then subtract Σc_0 and optimize



Duality

- Concept:
  - –A "dual" is a mirror-image form to another problem (the "primal")
  - –If primal = max; then dual = min
    If primal = min; then dual = max
  - –Dual contains all information of the primal, but in a different format
  - –Optimum value of primal = optimum value of dual
- Example:
  - –Primal: maximize output subject to budget limitations
  - –Dual: minimize costs subject to output requirements

LP Duality

- **Mathematics:**
  - Given a Primal:
    - Optimize: $Z = \underline{c}\,\underline{X}$
    - subject to: $\underline{A}\,\underline{X} \underline{<} \underline{>} \underline{B}$
  - Dual is:
    - Optimize: $Y = \underline{B}^T\underline{W}$
    - subject to: $\underline{A}^T\underline{W} \underline{<} \underline{>} \underline{c}^T$
- **Change of dimensionality between primal & dual:**
  - $\underline{c}^T$ and $\underline{B}$ have different number of variables
- **Can use duality to:**
  - Reduce size of constraint matrix
  - Speed up LP solution

LP Duality - Example

- Primal: Max: $Z = X_1 + 2X_2 + 3X_3$

  s.t. $4X_1 + 2X_2 \quad \underline{<} 5$

  $6X_1 + 7X2 + 9X_3 \underline{<} 12$

- A = 4 2 0 / 6 7 9 $\quad A^T =$ 4 6 / 2 7 / 0 9
- B = 5 / 12 $\quad B^T =$ 5 12
- C =1 2 3 $\quad C^T =$ 1 / 2 / 3

LP Duality - Example - continued

–Primal: Max: $Z = X_1 + 2X_2 + 3X_3$

s.t. $4X_1 + 2X_2 \leq 5$

$6X_1 + 7X2 + 9X_3 \leq 12$

–Dual: Min: $Y = 5W_1 + 12\,W_2$

s.t. $4W_1 + 6W_2 \geq 1$

$2W_1 + 7W_2 \geq 2$

$9W_2 \geq 3$

LP Duality - Interpretation of Results

–Primal:

Max: $Z = 3X_1 + X_2 + 8X_3$

s.t. $X_1 + X_3 \leq 4$

$X_1 + X_2 + X_3 \leq 7$

$2X_2 + X_3 \leq 8$

X*	= {0,2,4}
SP*	= {7.5,0,0.5}
OC*	= {4.5,0,0}
SV*	= {0,1,0}
Z*	= 34

–Dual:

Min: $Y = 4W_1 + 7\,W_2 + 8\,W_3$

s.t. $W_1 + W_2 \geq 3$

$W_2 + 2W_3 \geq 1$

$W_1 + W_2 + W_3 \geq 8$

W*	= {7.5,0,0.5}
dSV*	= {4.5,0,0}
dSP*	= {0,2,4}
dOC*	= {0,1,0}
Y*	= 34

Dual/Primal Solution Relationships

Primal		Dual
Decision Variables	→	Shadow Prices
Shadow Prices	→	Decision Variables
Opportunity Costs	→	Slack Variables
Slack Variables	→	Opportunity Costs