

Linear Programming Sensitivity Analysis

Engineering Systems Analysis for Design
Massachusetts Institute of Technology

Richard de Neufville, Joel Clark and Frank R. Field
LP Sensitivity Analysis Slide 1 of 19

Sensitivity Analysis

- **Rationale**
- **Shadow Prices**
 - Definition
 - Use
 - Sign
 - Range of Validity
- **Opportunity Costs**
 - Definition
 - Use

Engineering Systems Analysis for Design
Massachusetts Institute of Technology

Richard de Neufville, Joel Clark and Frank R. Field
LP Sensitivity Analysis Slide 2 of 19

Rationale for Sensitivity Analysis

- **Math problem is an approximation**
 - optimum is an approximation
 - we need to check
- **Constraints often artificial**
 - Designer should question them
 - *Should we have different specifications?*
- **Situations always probabilistic**
 - Prices change
 - Need to assess risk

Shadow Price Definition

- **Recall from Constrained Optimization:**
 - **Shadow price = δ (objective function) / δ (constraint) at the optimum**
 - **Complementary Slackness:**
Either (Slack variable) or (shadow price) = 0

Shadow Price Illustration

$$\begin{aligned} \text{Max: } & Y = X_1 + 4X_2 \\ \text{s.t. } & X_1 + X_2 \leq 5 = b_1 \\ & X_1 \geq 3 = b_2 \\ & X_2 \leq 3 = b_3 \\ & X_1, X_2 \geq 0 \end{aligned}$$

Notes:

a) $X_1^* = 3$; $X_2^* = 2$; $Y^* = 11$

b) when $\Delta b_1 = \pm 1$

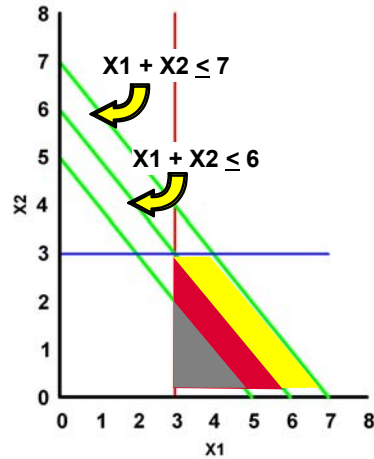
$$\Delta X_2^* = \pm 1; \Delta Y^* = \pm 4; SP_1 = 4$$

c) $SP_3^* = 0$; $\text{slack}_3 = 1$

d) when $b_1 > 6$

$$\text{slack}_3 = 0; SP_3 \neq 0;$$

$$SP_1 = 1 \leq 4$$



Proactive Use of Shadow Prices

- Identify constraints with high S.P
- See if they can be changed for better solutions
- Example: New York water supply
 - Original Design for Third City Tunnel (\$1 billion plus)
 - pressure < 40 psi at curb (some point in Brooklyn)
 - No allowance for local tanks, pumps
 - Shadow price in millions of dollars!

Reactive Use of Shadow Prices

- Respond to new opportunities
- Example: client changes specifications

- Respond to proposals for new constraints
- Example: trace chemicals

Sign of Shadow Prices

- "Obvious Rule" (+SP with $+\Delta b$) not correct
- Correct Reasoning:
 - What makes the optimum better?
 - Expansion of feasible region => "Relaxation of constraints"
 - What changes will increase the feasible region?
 - Increase upper bound $\sum_j a_{ij}X_j < b_i$
 - Decrease lower bound $\sum_k a_{kj}X_j > b_k$
 - i.e., "Raise the roof, lower the floor."

Shadow Price Illustration

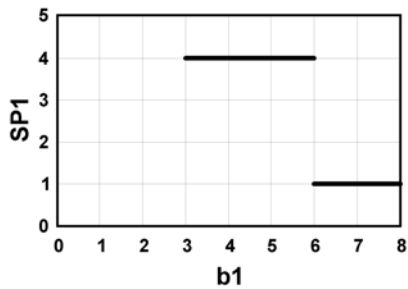
Max: $Y = X_1 + 4X_2$
s.t. $X_1 + X_2 \leq 5 = b_1$
 $X_1 \geq 3 = b_2$
 $X_2 \leq 3 = b_3$
 $X_1, X_2 \geq 0$

Notes:

- a) $X_1^* = 3; X_2^* = 2; Y^* = 11$
- b) when $\Delta b_1 = \pm 1$
 $\Delta X_2^* = \pm 1; \Delta Y^* = \pm 4; SP_1 = 4$
- c) $SP_3^* = 0; \text{slack}_3 = 1$
- d) when $b_1 > 6$
 $\text{slack}_3 = 0; SP_3 \neq 0;$
 $SP_1 = 1 \leq 4$



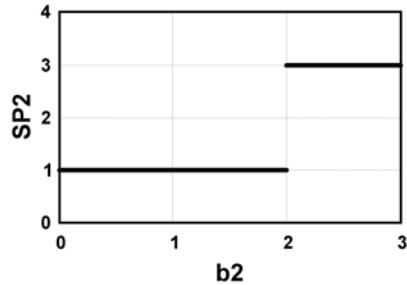
Shadow Prices As Constraints Change



decrease a lower bound
("lower the floor")

$b_2: 3 \rightarrow 2$
 new $X^* = [2, 3]$
 new $Y^* = 14$
 $\Delta Y^* = 3$

increase an upper bound
("raise the roof")

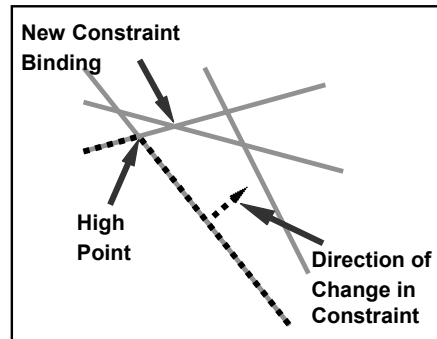


Range of Shadow Prices

- In Linear Programming, Shadow prices are constant
- Until a constraint changes enough so that a new constraint is binding
- Results given as:

$$SP_K = \text{constant} \quad \text{for } r_L < b_K < r_U$$

- Outside the range:
 - Shadow prices decrease as constraint is relaxed
 - Shadow prices increase as constraint is tightened



Shadow Price Ranges for Example

$$\begin{aligned} \text{Max: } & Y = X_1 + 4X_2 \\ \text{s.t. } & X_1 + X_2 \leq 5 = b_1 \\ & X_1 \geq 3 = b_2 \\ & X_2 \leq 3 = b_3 \\ & X_1, X_2 \geq 0 \end{aligned}$$

Shadow Prices

$$\begin{aligned} SP_1 = 4 & \quad 3 \leq b_1 \leq 6 \\ SP_2 = 4 & \quad 2 \leq b_2 \leq 5 \\ SP_3 = 0 & \quad 2 \leq b_3 \end{aligned}$$



Opportunity Cost - Definition

- Objective Function = $\sum c_i X_i$
- Opportunity costs associated with c_i -- the coefficients of design/decision variables
- At optimum, some decision variables = 0
 - These are non-optimal decision variables
- Opportunity cost is:
 - Degradation of optimum per unit of non-optimal variable introduced into design
 - A "cost" in that it is a **worsening** of optimum. Units may be almost anything; equal to whatever units are being optimized.

Meaning of Opportunity Costs

- Opportunity cost defines design trigger "price"
 - The value of the coefficient of the decision variable for which that variable should be in the design
- Suppose: Obj.Function = ... + $c_K X_K$ + ...
and X_K not optimal with an opportunity cost = OC_K
- Then, as c_K changes for the better, (greater for maximization, lesser for minimization)
 - OC_K lower
 - $OC_K = 0$ at $c_K' = c_K - OC_K$
- c_K' is trigger price; defines the limit of best design

Illustration of Opportunity Cost

- What happens when forced to use a non-optimal decision variable?
 - Example: $\text{Min Cost} = 2X_1 + 10X_2 + 20 X_3$

$$\text{s.t.} \quad X_1 + X_2 + X_3 \geq 3$$

$$X_2 \geq 1$$

$$X_1, X_2, X_3 \geq 0$$
 - $\underline{X}^* = (2, 1, 0)$; $\text{cost}^* = 14$
 - If forced to use X_3 , new $X^* = (1, 1, 1)$; new $\text{cost}^* = 32$
- Thus: $(\text{opportunity cost})_3 = \Delta Z^*/1 = 18$

Use of Opportunity Cost

- At what price would it be desirable to use X_3 ?
- If X_3 is used with no change in its unit cost ($= c_3$), the optimal cost would increase by 18
- If the cost of X_3 were to fall by an amount equal to the opportunity cost ($c_3' = c_3 - \text{OC}_3 = 20 - 18 = 2$). It would then compete with X_1
- So the answer is: When its unit cost falls by its opportunity cost: $20 - 18 = 2$

How do you find SP and OC?

- LP optimization programs all calculate shadow prices and opportunity costs routinely and “print them out” for you
- Sometimes, programs report this information in special ways. Thus:
 - Shadow Prices \Leftrightarrow “dual decision variables”
 - Opportunity Costs \Leftrightarrow “dual slack variables”
 - More on this later

A Possible Semantic Confusion

- Note that the Phrases “shadow price” and “opportunity cost” have somewhat different meanings in LP and Economics literature
- The “opportunity cost” of an action in economics can be interpreted as the “shadow price” of that action on the budget...

Summary on LP Sensitivity Analysis

- **LP Optimization Programs automatically provide important information useful for improving/changing design**
- **Shadow prices -- to help redefine constraints**
- **Opportunity costs -- to identify critical prices**

- **Need to understand these quantities carefully**