Introduction to Linear Programming (LP)

- Mathematical Programming Concept
- LP Concept
- Standard Form
- Assumptions
- Consequences of Assumptions
- Solution Approach
- Solution Methods
- Typical Formulations

Mathematical Programming (MP) Concept (1)

- Definition
  - M.P. includes a range of powerful computer-based optimization methods
- Approach to Optimization
  - M.P methods exploit peculiar features of the structure of a problem to get solutions efficiently
  - Different M.P. methods exploit different structures or features
  - Understanding which features apply to a particular problem -- and thus which method can apply -- is important

Mathematical Programming (MP) Concept (2)

- Two Large Categories of Methods
- Those Valid for Convex Feasible Regions
  - Linear Programming, etc
  - These use a local search routine to reach global optimum
  - "Keep going up/down until reach top/bottom"
  - Very efficient -- but not always valid
- Those Valid for Non-Convex Feasible Regions
  - Dynamic programming, etc
  - These "enumerate" solutions to discover optimum
  - They "prune," that is, eliminate, possible solutions because these can be shown to be "dominated"
  - Computations limit applicability to special situations, but good for options analysis

LP Overview

- Special form of mathematical programming
  - Equations must be linear
- Uses simple solution procedures
  - Linear algebra
- Very powerful
  - Extremely large problems
  - 100,000 variables
  - 1000's of constraints
  - Note difference!
- Useful design information through Sensitivity Analysis
  - Answers to "what if" questions

Standard Form of LP - Three Parts

- Objective function
  - maximize or minimize

  X_i known as decision variables

  - $Y = \Sigma\ c_i x_i$
  - $Y = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

- Constraints
  - subject to:

  - $a_{11}x_1 + a_{12}x_2 + \ldots < = > b_1$
  - $a_{21}x_1 + a_{22}x_2 + \ldots < = > b_2$
  - $a_{31}x_1 + a_{32}x_2 + \ldots < = > b_3$

- Non-Negativity

  - $x_i \geq 0$ for all i

Standard Form of LP -- Summary

Optimize $Y = \underline{c}\ \underline{X}$

subject to: $\underline{A}\ \underline{X}$ (< or = or >) $\underline{b}$

$X \geq 0$

Three LP Assumptions (1)

- Linearity
- Additivity
- Non-Negativity

- **Linearity of Objective Function and Constraints**
  - Essential Condition is:

    $f(k\underline{X}) = k\ f(\underline{X})$

  - for example: $f(X) = 3 + 4X_1 + 2X_2$
    is NOT linear in the LP sense

- **Implies**
  - Constant returns to scale (only first order terms)
  - No "fixed charges" (no constants)

Three LP Assumptions (2)

- **Additivity:**

  $f(X_1, X_2, \ldots, X_n) = f(X_1) + f(X_2) + \ldots + f(X_n)$

  - no interactive effects among Xi terms
  - assumes that individual segments of the problem operate as well independently as together

- **Non-Negativity**

  $X_i > 0$

  - no fundamental difficulties except in particular situations

Consequences of Assumptions

- Convexity of feasible region (if it exists!)
- Convex feasible region, with linear objective function, implies:
  - –Optimum will be on an edge of the feasible region
- Since edges are also linear
  - –Optimum is at a corner point
    (can be several in special cases)
- Note that corner points
  - –Constitute small, finite set
  - –Defined by solution of linear equations

<u>Bottom Line:</u> **Assumptions imply the existence of an efficient solution strategy**

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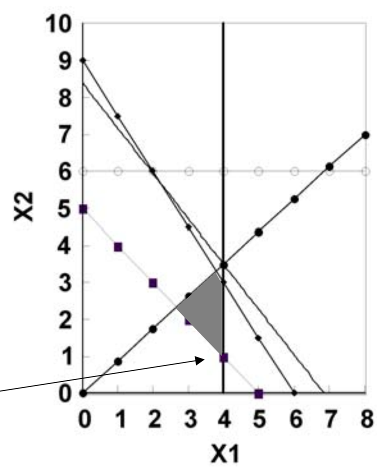
Example for Assumptions

minimize: $Z = 3X_1 + 5X_2$

s.t.
$$X_1 + X_2 \geq 5$$
$$3X_1 + 2X_2 \leq 18$$
$$6X_1 + 5X_2 \leq 42$$
$$-7X_1 + 8X_2 \leq 0$$
$$0 \leq X_1 \leq 4$$
$$0 \leq X_2 \leq 6$$

$X_1^* = 4$
$X_2^* = 1$



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Solution Approach

- Find a corner point
  - An "initial feasible solution"
- Proceed to improved corner points
- Stop when no further improvements are possible

Solution Calculations

- To find a corner point
  - it is necessary to solve system of constraint equations
  - from linear algebra, this requires working with matrix of constraint equations, specifically, manipulating the “determinants”
  - Amount of effort set by number of constraints
- Thus, number of constraints defines amount of effort
- This is why LP can handle many more decision variables than constraints

Solution Methods

- **Simplex**
  - –**The textbook method**
  - –**For step 2, select improved corners**
    - –*Always goes to best corner*
    - –*Searches until no further improvement possible*
  - –**Inefficient for real problems**
  - –**Not used in practice**
- **Practical methods - many exist - often proprietary**
  - –**Step 2 takes many forms**
  - –**Each best for different cases**
  - –**Very great efficiency possible**
  - –**A real art!**

Excel LP Methods

- **Several LP methods available for simple problems**

- What's Best http://www.lindo.com/cgi-bin/
- Solver an Excel tool

- **These methods use standard equation and need no special organization**
- **They provide all kinds of sensitivity information**
- **The difficult part is setting up the LP equations so that they make sense!**

Typical Formulations: "Transportation" Problem

Objective = Minimize cost of moving a single commodity from sources "i" to uses "j"

$= \Sigma C_{ij} X_{ij}$

subject to:

Amount shipped < Amount available $\Sigma_j X_{ij} \leq S_i$

Amount delivered > Uses $\Sigma_i X_{ij} \geq S_j$

Note:

Matrix of constraint coefficients are all 0's and 1's

- Particularly efficient solutions

Typical Formulations: "Blending" or "Diet" Problems

Objective = Minimize cost of materials

$= \Sigma C_i X_i$

subject to:

Limits on availability $X_i \leq$ Amounts given

Maxima or minima on impurities, trace elements, nutritional requirements, etc... $\Sigma a_{ij} X_i \leq = \geq b_j$

Example:

- Minimize cost of steel alloy when only so much scrap is available, subject to limitations on carbon content, trace elements, etc.

Typical Formulations: "Activity" Problems

Objective = Minimize cost of production

$= \Sigma C_i X_i$

where X_i represent "activities", that is, specific ways or fixed ratios of using resources

subject to:

Limits on resources $\Sigma a_{ij} X_i \leq = \geq b_j$

Example:

–Minimize cost of delivering cargo where each "activity" represents the use of a different size of ship, each with its own implications for the use of crew, fuel, etc.

Summary

- LP can handle very large problems
- Basic mathematics simple
- User can focus on definition of the problem
- Realistic problems require special techniques to deal with non-linearities, integral variables, etc. Can be very sophisticated
- This presentation is only an introduction!