Marginal Analysis Outline

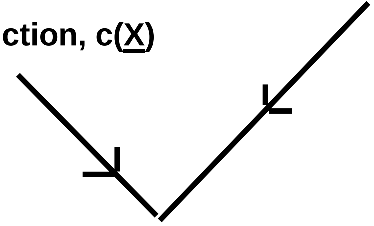
1. Definition and Assumptions
2. Optimality criteria
   - Analysis
   - Interpretation
   - Application
3. Key concepts
   - Expansion path
   - Cost function
   - Economies of scale
4. Summary

Marginal Analysis

- Basic form of optimization of design
- Combines:

  Production function - Technical efficiency

  Input cost function, c(<u>X</u>)



Economic efficiency

Assumptions of Marginal Analysis

- Feasible region is convex (over relevant portion) This is key. Why?
- To guarantee no other optimum missed
- No constraints on resources
- Models are “analytic” (needed for proof)
- Defines optimum by looking at the margins -- the derivatives

Optimality Conditions for Design, by Marginal Analysis

The Problem:

Min $C(Y') = c(\underline{X})$ cost of inputs

s.t. $g(\underline{X}) = Y'$ production function

vector of resources

Fixed level of output

The Lagrangean:

$$L = c(\underline{X}) - \lambda\,[g(\underline{X}) - Y']$$

Optimality Conditions for Design, by Marginal Analysis (2)

- **Key Result:**

$$\partial c(\underline{X}) / \partial X_i = \lambda\, \partial g(\underline{X}) / \partial X_i$$

$\uparrow$ marginal cost $\uparrow$ marginal product

- **Optimality Conditions:**

$$MP_i / MC_i = MP_j / MC_j = 1 / \lambda$$

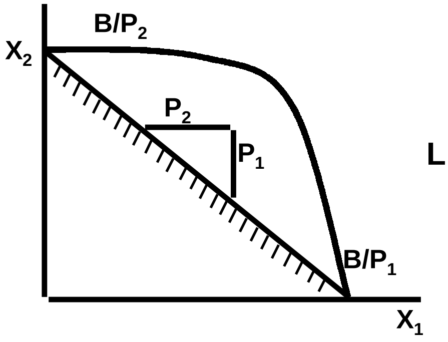
$MC_j / MP_j = MC_i / MP_i = \lambda$ = Shadow Price on Product

- A balanced design

  Each X_i contributes "same bang for buck"

Graphical Interpretation of Optimality Conditions

(A) Input Cost Function

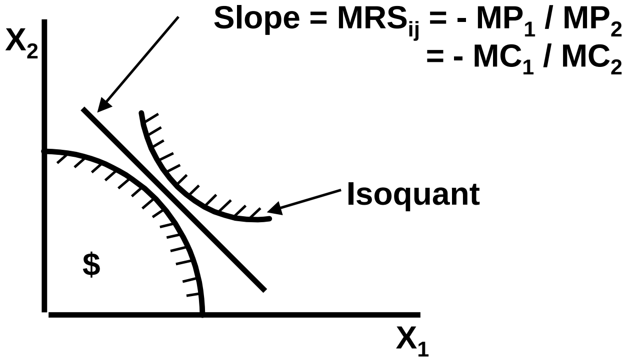


B = Budget
$c(\underline{X}) = \Sigma p_i X_i \leq B$

Linear case:
In general, non-linear
(as in curved line)

Graphical Interpretation of Optimality Conditions (2)

(B) Conditions



Slope = $MRS_{ij} = - MP_1 / MP_2$
$= - MC_1 / MC_2$

Application of Optimality Conditions

Problem: $Y = a_0 X_1^{a1} X_2^{a2}$
$c(\underline{X}) = \Sigma p_i X_i$

Note: Linearity of Input Cost Function
- typically assumed by economists
- in general, not valid
  - prices rise with demand
  - wholesale, volume discounts

Solution:

$$[a_1 / X_1^*]\ Y / p_1 = [a_2 / X_2^*]\ Y / p_2$$

(* denotes an optimum value)

Expansion Path

- Locus of all optimal designs $\underline{X}^*$
- Not a property of technical system alone
- Depends on local prices
- Optimal designs do not, in general, maintain constant ratios between optimal X_i^*

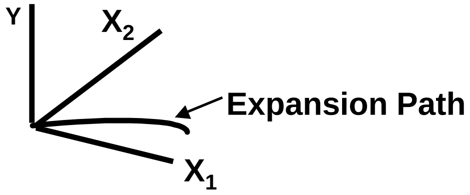
e.g.: crew of 20,000 ton ship
crew of 200,000 ton ship

Calculation of Expansion Path

- Assume: $Y = 2X_1^{0.48} X_2^{0.72}$ (increasing RTS)
  $c(\underline{X}) = X_1 + X_2^{1.5}$ (increasing costs)
- Optimality Conditions:

$$(0.48 / X_1)\ Y / 1 = (0.72 / X_2)\ Y / (1.5X_2^{0.5})$$
$$= MP_i / MC_i$$
$$\Rightarrow X_1^* = (X_2^*)^{1.5}$$

- Graphically:

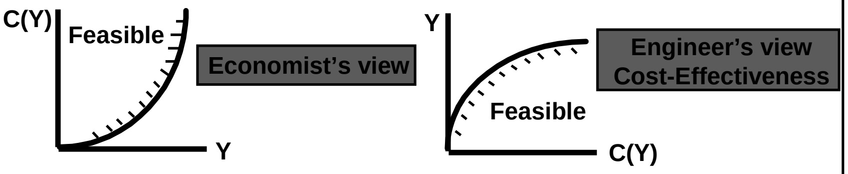


Cost Functions

- **Not same as input cost function**
  - It represents the optimal cost of Y
  - Not the cost of any set of <u>X</u>
- **C(Y) = C(<u>X</u>*) = *f* (Y)**

Cost Functions (2)

- **Graphically:**



- **Great practical use:**
  - How much Y for budget?
  - ΔY for ΔB?
  - Cost effectiveness, $\Delta B / \Delta Y$

Calculation of Cost Function

- Cobb-Douglas Prod. Fcn: $Y = a_0 \pi X_i^{a_i}$

- Linear input cost function: $c(X) = \Sigma p_i X_i$
- Result
  $C(Y) = A(\pi p_i^{a_i/r})Y^{1/r}$ where $r = \Sigma a_i$

- Easy to estimate statistically
  => Solution for ‘a_i’
  => Estimate of prod. fcn. $Y = a_0 \pi X_i^{a_i}$

Calculation of Cost Function (2)

- Assume Again:
  $Y = 2X_1^{0.48} X_2^{0.72}$
  $c(\underline{X}) = X_1 + X_2^{1.5}$

- Expansion Path: $X_1^* = (X_2^*)^{1.5}$
  Thus: $Y = 2(X_2^*)^{1.44}$
  $c(\underline{X}^*) = 2(X_2^*)^{1.5}$
  => $X_2^* = (Y/2)^{0.7}$
  $c(Y) = c(\underline{X}^*) = (2^{-0.05})Y^{1.05}$

Economies of Scale

- A possible characteristic of cost function
- Concept similar to returns to scale, except
  - ratio of ‘X_i’ not constant
  - refers to costs (economies) not “returns”
- Economies of scale exist if costs increase slower than product

  Total cost = $C(Y) = Y^{\propto}$ $\propto < 1.0$

Economies of Scale (2)

- Note:

  If Cobb-Douglas, linear input costs

  Increasing RTS

  => Economies of scale

  $r = \Sigma a_i > 1.0$ => $C(Y) = \text{fcn } Y^{1/r}$

Not necessarily true in general

See example!!

$c(Y) = c(\underline{X}^*) = (2^{-0.05})Y^{1.05}$

Marginal Analysis Summary

- **Assumptions --**
  - -- convex feasible region
  - -- Unconstrained
- **Optimality Criteria**
  - – MC/MP same for all inputs
- **Expansion path -- Locus of Optimal Design**
- **Cost function -- cost along Expansion Path**
- **Economies of scale (vs Returns to Scale)**
  - -- Exist if Cost/Unit decreases with volume