Valuation of Options:
Theory

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Outline

- Payoffs from options
- Influences on value of options
  - Value and volatility of asset ; time available
- Basic issues in valuation: risk aversion
- Alternative approaches to valuation
  - Decision analysis vs. Options analysis
- Valuation by replication
- Black-Scholes equation
- Generalized binomial
- Crucial role of risk-free discount rate
- Summary

Uncovering the Sources of Value in Options

- Working toward placing an exact value on options

- Need to build up to valuation
  - Identify interesting features
  - Examine influences of value
  - Combine findings into valuation framework

- Start by looking at payoffs from options
  - Payoff structure influences value
  - Payoffs and value are however different

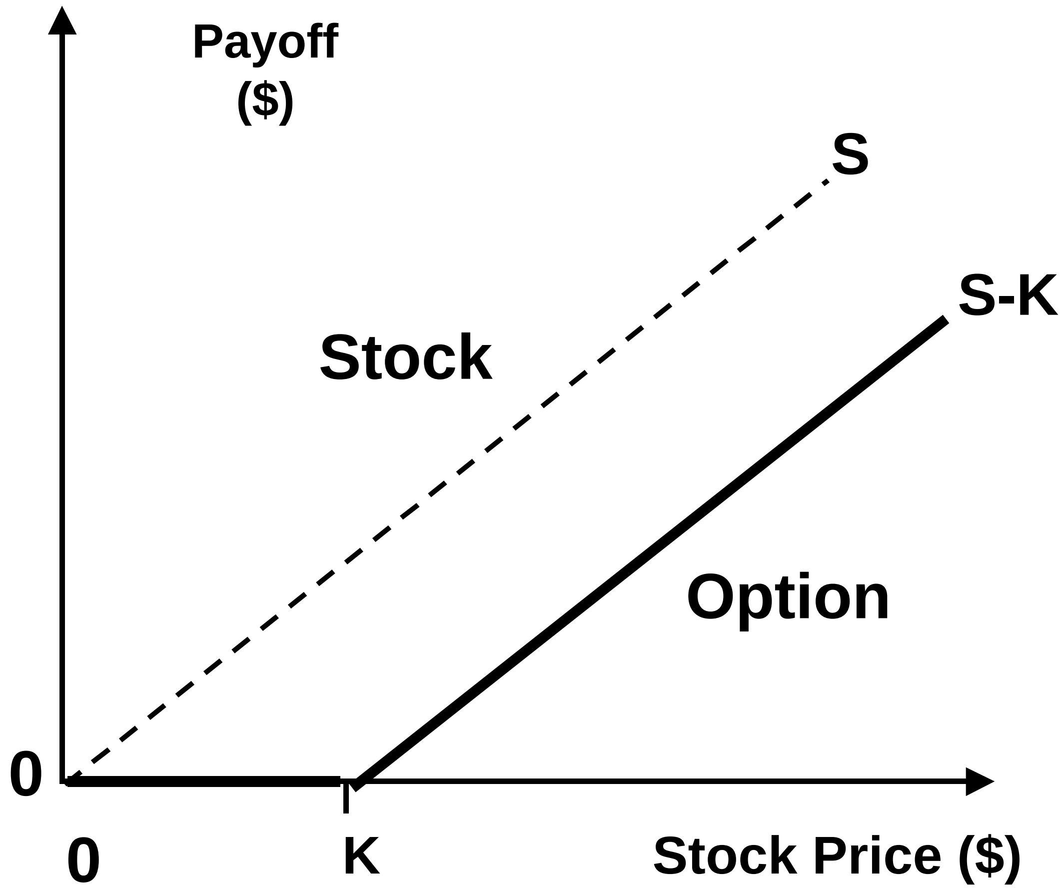
Recall Definitions for Options

- S = stock price at any time
- S* is price at time you exercise option
- K = strike price at which stock can be bought (call) or sold (put)
- T = time remaining until option expires
- ß = standard deviation of returns for stock (volatility)
- R = risk-free rate of interest

Call Option Payoff

- If exercised, call option owner buys stock for a set price
  - Get stock worth S* dollars
  - Pay strike price of K dollars
  - Net position = S*–K
- If unexercised, net payoff is zero
- Maximum of either 0 or S*–K = net payoff for call
- Net payoff for call = max [0, S*–K]

Payoff Diagram for Call Option



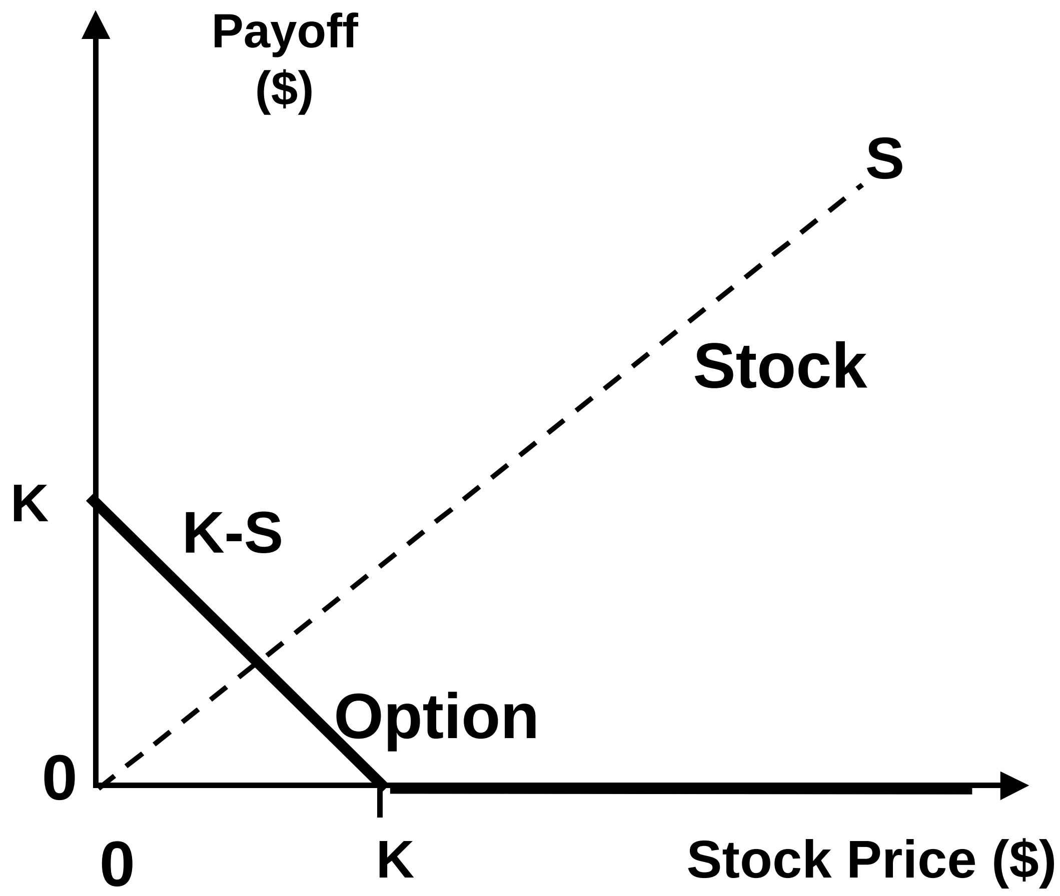
Put Option Payoff

- If exercised, put option owner sells stock for a set price
  - Sell stock worth S* dollars
  - Receive strike price of K dollars
  - Net position = K–S*

- If unexercised, net payoff is zero

- Net payoff for put = max [0, K–S*]

Payoff Diagram for Put Option



Valuation of Options

- How much should you pay to acquire an option?

- Payoff diagrams show for a given strike price
  - Call payoff increases with stock price
  - Put payoff decreases with stock price

- Immediate payoff may not reflect full value of option
  - Owner exercises only when advantageous
  - Must compare immediate exercise value with waiting

Why immediate payoff and value might differ

- **Consider an at the Money Option (S=K)**

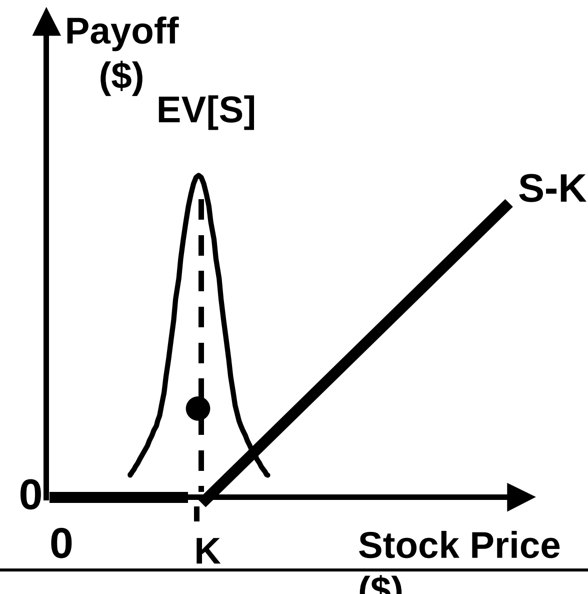
  Immediate Exercise Payoff Is Zero

  Positive Payoff Might Be Obtained by Waiting

  Worst Outcome of Waiting Is Zero Payoff (Same As Immediate Exercise)

Value in Ability to Wait Not Reflected in Immediate Exercise

● = value of option



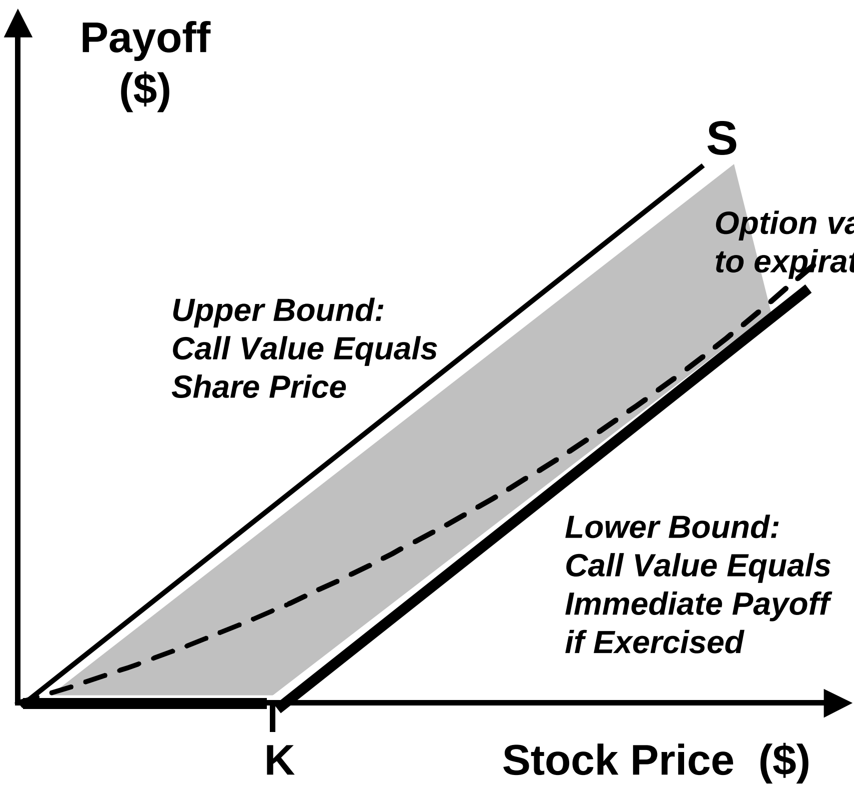
Narrowing the scope: boundaries on price

- **Some Logical Boundaries on the Price of an American Call**

Price $\geq$ 0
Otherwise Buy Option Immediately

Price $\leq$ S
Stock Yields S*
Option Yields S*- K
Option Worth Less Than Stock

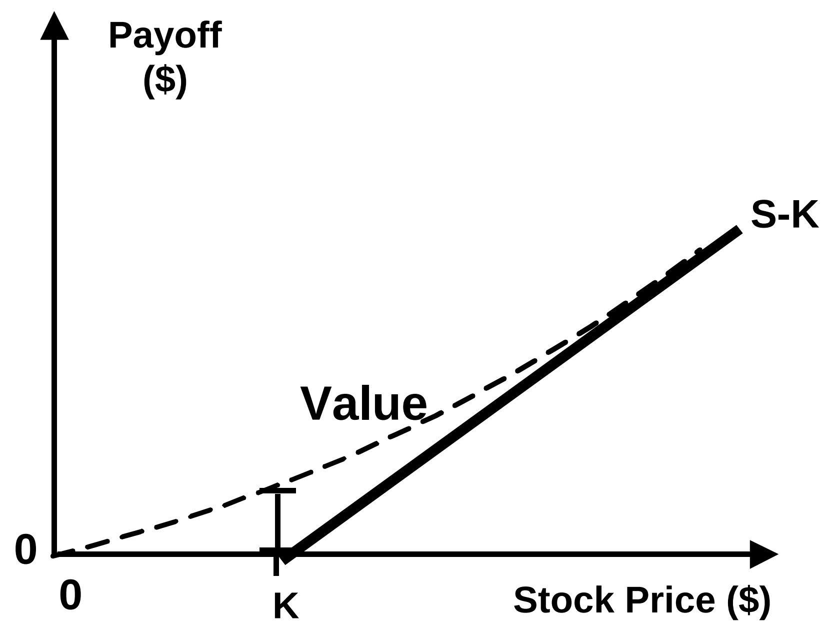
Price $\geq$ S - K
Or Buy and Exercise Immediately



Examining Value for All Stock Prices I

- Value exceeds immediate exercise payoff

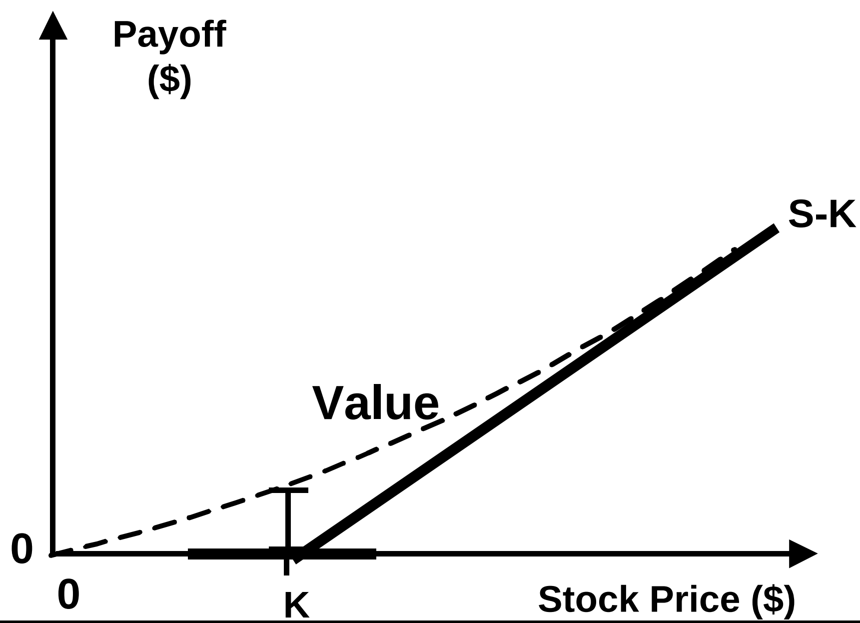
- Asymptotically approaches payoff for increased S
  - Incentive to lock in gain becomes significant



Examining Value for All Stock Prices II

- **Approaches zero as stock price nears zero**
  - **Option is worthless if stock reaches zero**

- **What influences difference between value & immediate payoff?**



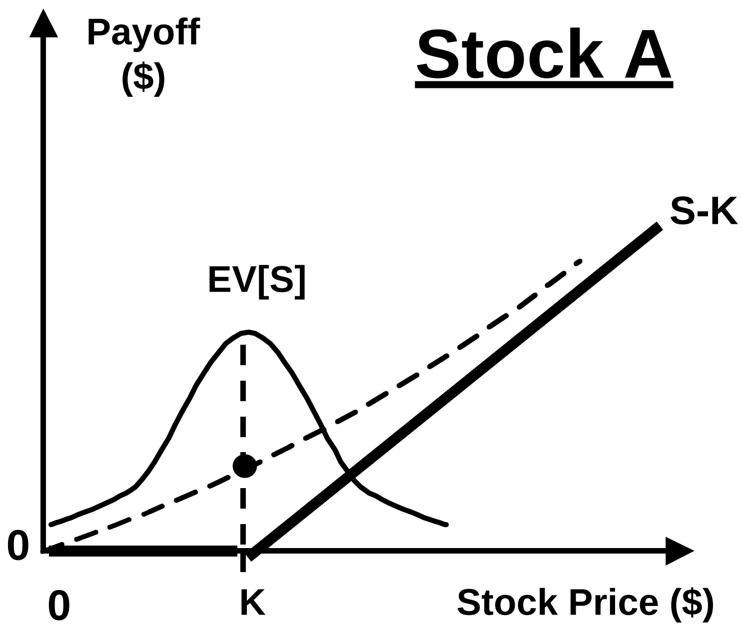
Impact of Time

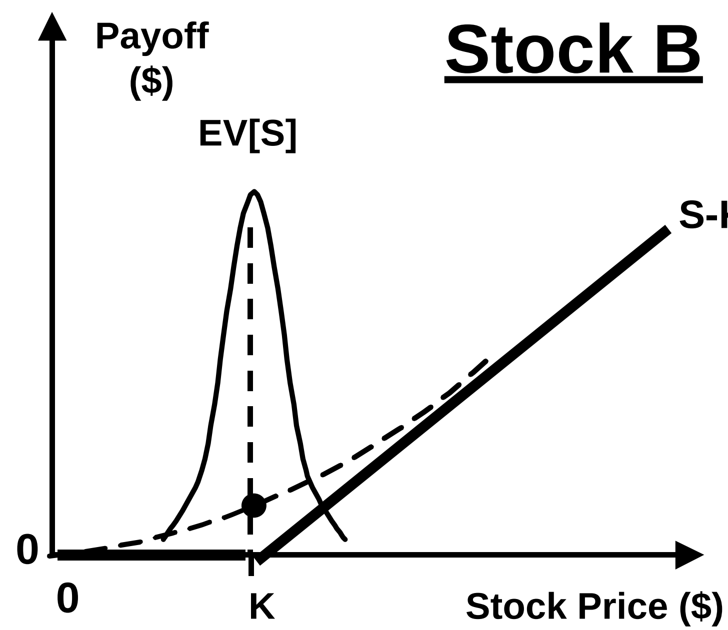
- **Increasing time to expiration increases option value**
  - Ability to wait allows option owner to benefit from asymmetric returns
  - Longer- term american option contains shorter-term options, plus more time
- **Compare a 3 and 6 month american call**
  - Can exercise 6 month call at same time as 3 month
  - Can wait longer with 6 month
  - Which is more valuable?
- **Time impact less clear for european options**
  - Forced to wait to exercise
  - Could miss out on profitable opportunities

Option value increases with volatility

- **Two at the Money Options (S=K)**
  Both Have 50% Chance of Zero Payoff
  Underlying With Greater Volatility Has More Opportunity for Large Payoffs

Asymmetric Returns Favor High Variation (Limited Losses)

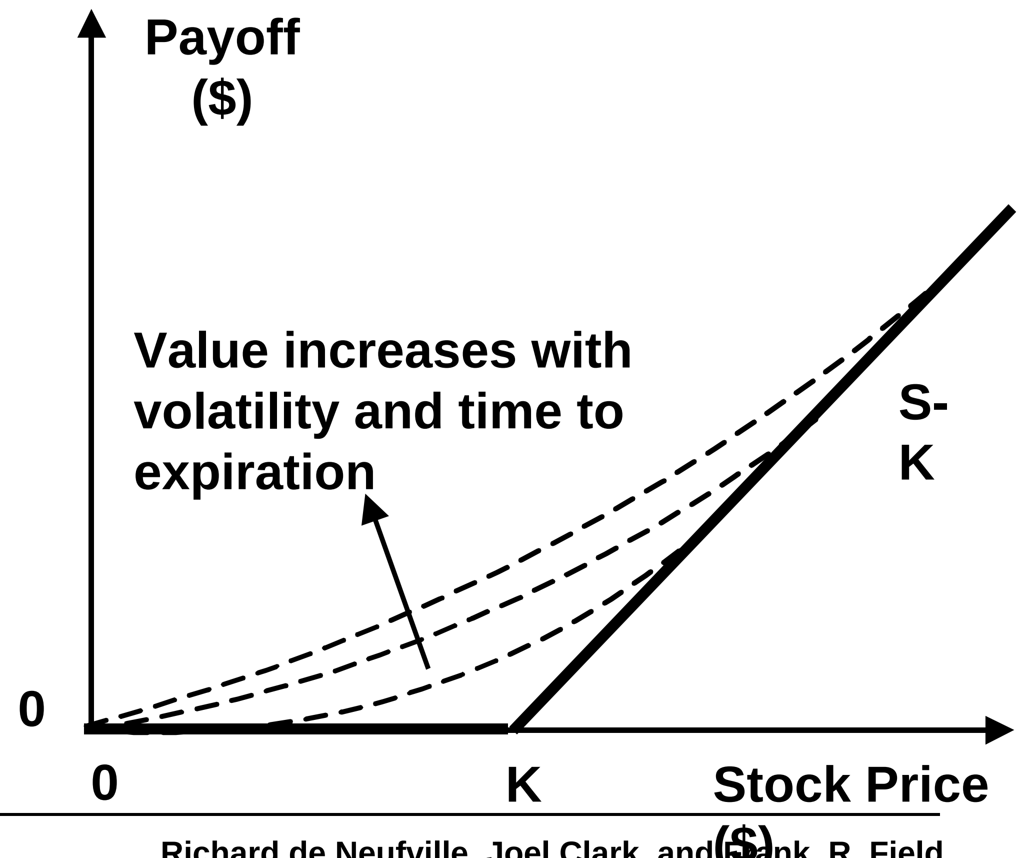




Generalized American Call Option Value

- For a set strike price, call option value increases with
  - Stock price increases
  - Volatility
  - Time

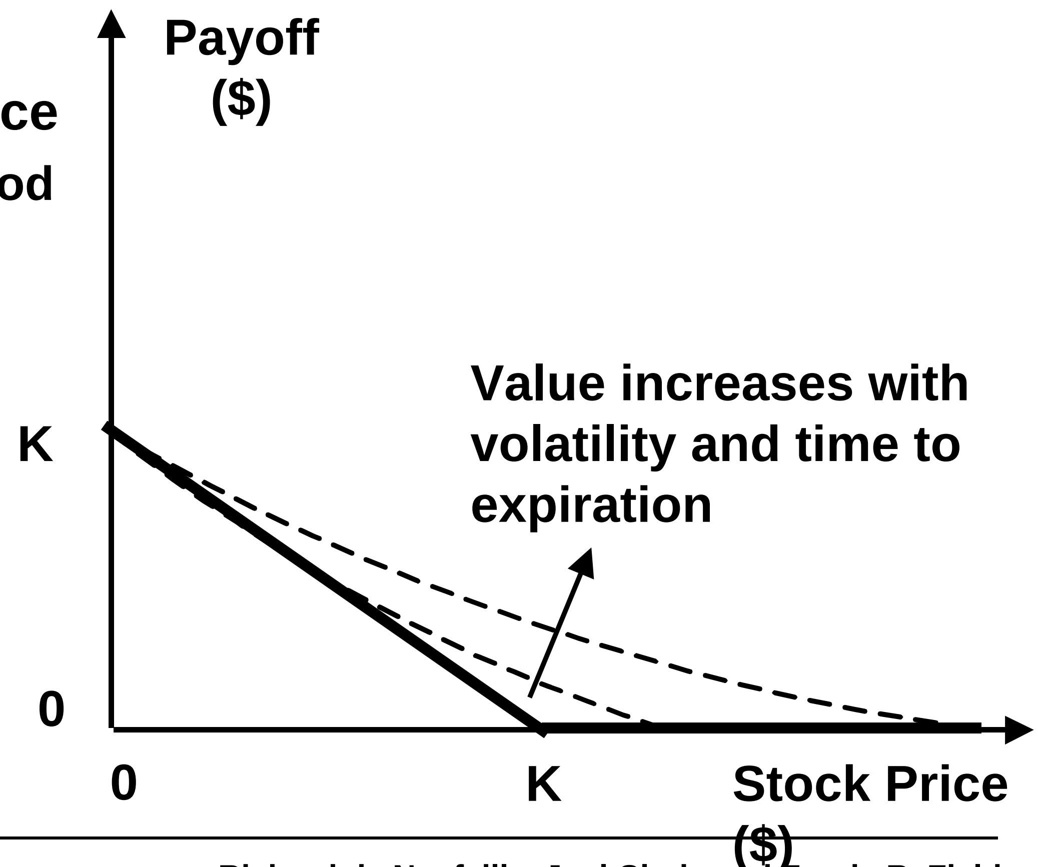
- Increased strike price
  - Reduces likelihood of payoffs
  - Reduces call option value



Generalized American Put Option Value

- For a set strike price, put option value increases with
  - Stock price declines
  - Volatility
  - Time

- Increased strike price
  - Increases likelihood of payoffs
  - Increases put option value



Summary Influences of Value of Options

- Payoffs of options
- Value increases with value of asset, time available
- Value of option increases with volatility
  - More risk => more value

- Increase in value with volatility is key point,
- Counterintuitive to most people
- Intuitive explanation: insurance is more valuable when risk is greater

Basic Issue in Valuation: Risk Aversion

- **Risk aversion phenomenon**
  - People value results non-linearly (utility = $ exp a)
  - E.G.: More than $1000 of gain required to balance $1000 loss
  - Equivalent to risk aversion
- **Utility is one way to reflect this phenomenon**
- **CAPM is alternate way**
  - Discount rate increases with risk
  - Projects with more risk (possibility of loss) have to have higher returns
- **Each method has its advantages**
  - CAPM deals best with financial risks
  - Utility best to deal with non-financial aspects

Alternate Ways to Deal With Risk Aversion

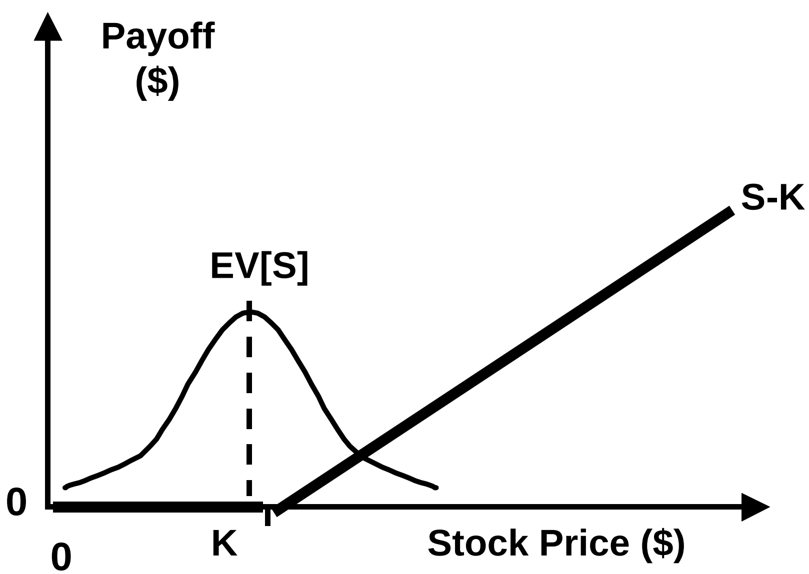
- Two ways to handle this for valuation
  - Two parameters that can be varied:
  - Probability of events
  - Amount of outcome
- Decision analysis works on outcome
  - Probabilities left alone
  - Amount of outcome transformed to utility of outcome
- Options analysis works on risk and discount rate
  - Discussion of procedure later

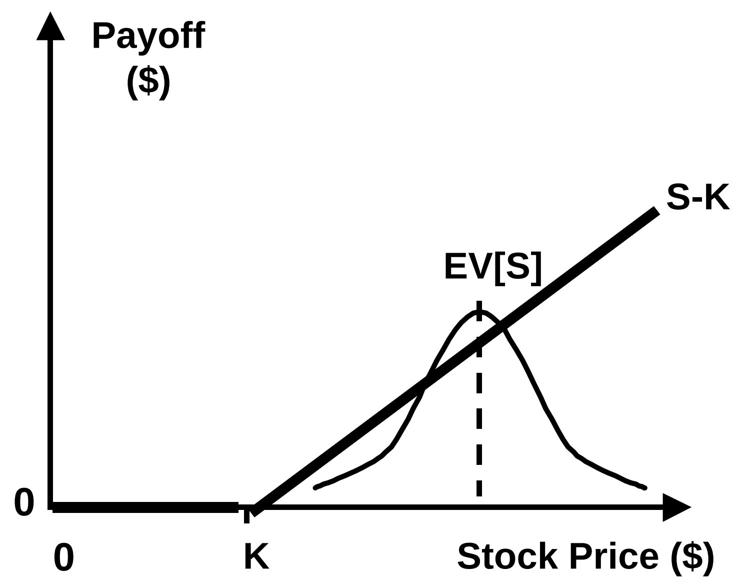
Deficiencies of Decision Analysis for Valuation of Options

- **Practical inability to handle market risks**
  - Prices vary rapidly, up and down
  - Excessive number of paths (e.G.: Dual fuel burner)
- **Theoretical issue: what discount rate?**
  - Should use discount rate adjusted for risk (CAPM), BUT
  - Stock prices change continually and unpredictably
  - Option risk changes with stock price
  - Cannot predict option risk over time
  - No single rate that applies
- **Options methods deal with variation of risk**
- **Option approach better when practical (not always for real systems)**

Why option risk changes unpredictably
Call option example

- **Payoff Becomes More Certain With Increased S**
  Possibility of Losing Entire Investment Decreases
  Decreases Volatility (Risk)





- **Risk of Option Changes When Stock Price Changes**
- **Stock Price Changes Continually and Unpredictably**

Valuation by Replication

- One approach is to replicate options payoffs using other assets
  - If end payoffs are the same, then
  - The initial value of these assets and the option should be equal
- Essential idea: an option implicitly involves 2 actions
  - Call option: like buying a stock with borrowed money
  - Put option: like selling stock with borrowed stock
- Key is to find exact replicating assets that can be valued directly

Replicating a Call Option

- If exercised, call option results in stock ownership
  - Option owner effectively controls shares of stock
- Payment for stock delayed until option is exercised
  - Delayed payments are essentially loans
- Call options are like buying stock with borrowed money
- Use this analogy to develop estimate of option value

A One-period Example (Call Option)

- Stock
  - Current price = $100
  - Price at end of period either $80 or $125
- One-period call option
  - Strike price = $110
- Assume funds can be borrowed at risk-free rate
  - One-period risk-free rate = 10%
- Identify conditions where end-of-period payoffs are equal
  - Buying stock and borrowing money
  - Buying call options
- Then, initial values should be equal

Call Option: Cost and Payoffs

- Pay C dollars to acquire option
- If S>K, call payoff = S - K
- If S<K, call payoff = 0

	Start (Stock = 100)	End (Stock = 80)	End (Stock = 125)
Buy Call Strike = 110	- C	0	(125 - 110) = 15

Stock Buy and Loan: Cost and Payoffs

- Buy stock and borrow to have payoffs equal option
- If S>K, stock and loan payment to net positive return
  - Find ratio so stock and loan payments equal option returns
- If S<K, want stock and loan payment to net to zero

	Start (Stock = 100)	End (Stock = 80)	End (Stock = 125)
Buy Stock	-100	80	125
Borrow Money	80/(1+r)	- 80	- 80
Net	-100 + 80/(1+r)	0	45

Comparing costs and payoffs

- **If S>K, Stock and Borrowing Returns More Than Call**
  Ratio of Returns in This Case Is 3:1

  If S<K, Returns Are Equal
  Buying 3 Calls Should Equalize Payoffs

	Start (Stock = 100)	End (Stock = 80)	End (Stock = 125)
Buy Call (Strike = 110)	- C	0	(125-110) = 15
	Start (Stock = 100)	End (Stock = 80)	End (Stock = 125)
Buy Stock and Borrow	-100 + 80/(1+r)	0	45

Equalizing Costs and Payoffs

- **Equal payoffs suggest initial costs should be equal**
  - **Otherwise could buy cheaper alternative and sell more expensive result would be instant profit**

	Start (Stock = 100)	Start (Stock = 80)	End (Stock = 125)
Buy 3 Calls Strike = 110	-3C	3*0 = 0	3*(125 - 110) = 45

	Start (Stock = 100)	Start (Stock = 80)	End (Stock = 125)
Buy Stock and Borrow	100+80/(1.1)	0	45

- **3C = -100 + 80/(1.1), therefore C = $9.09**

One-period Example Summary

- Call option payoff replicated using stock and borrowing
  - Cost of loan and price of stock are known
  - Allows value of option to be assessed
- Information needed to determine call value
  - Stock price
  - Strike price
  - Time (one-period)
  - Volatility of stock (range of final prices)
  - Interest rate

Options Pricing Models

- **Concept of example important, must extend to be practical**
  - Multiple periods
  - Dividends or other ongoing returns from asset
- **Present two options valuation frameworks**
- **Black-Scholes**
  - Reasonably compact formula
  - Prices european calls only (assumes exercise can occur only at expiration)
  - Can be modified to include dividends
- **A more general binomial model**
  - Less limited in scope, more difficult to apply
  - Considers exercise at any time and dividends

Black-Scholes Options Pricing Formula I

- **The value of a *european* call on a *non-dividend paying stock***

$$C = S * n(d_1) - K * e^{-rt} * n(d_2)$$

S = current stock price
K = striking price
R = RISK-FREE rate of interest
T = time to expiration
σ = Standard deviation of returns on stock
N(x) = standard cumulative normal distribution
D_1 = $LN\ [S/(K * e^{-rt})]\ /\ (\sigma * \sqrt{t}) + (\sigma * \sqrt{t})$
D_2 = $d_1 - (\sigma * \sqrt{t})$

Black-Scholes Options Pricing Formula II

- **Note similarities to replicating example**
  - Same factors required
  - Volatility replaces stock outcomes from one-period example
  - Resembles replicating portfolio (buy stock and borrow)
- **Derivation complicated, not the focus here**

Origin of Black-Scholes Model

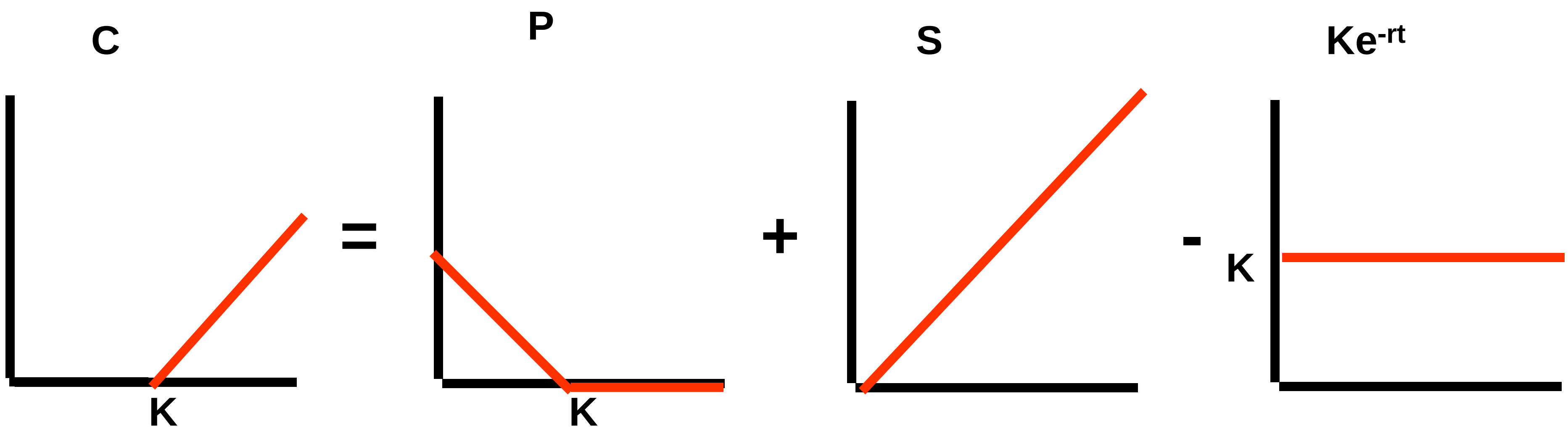
- One-period example
  - Compared end-of-period option value to stock and borrowing portfolio value
  - Equated beginning-of-period option value to initial portfolio value
- Black-Scholes model
  - Assumes many small periods
  - Represents limit as time period approaches zero
  - Calculates call option value based on statistically described stock movements
  - Assumes early exercise is not possible
- Needed for general model
  - Ability to decide to hold or exercise, at beginning of each period

Using Black-Scholes Model

- Essentially a substitution and solve formula
  - Programmed into most financial calculators
  - Ubiquitous to wall street community
- S, K, t are directly stated terms of option
- R is RISK-FREE discount rate of currency named in strike price
- Volatility of stock must be estimated from Historical data

A Relationship Between Calls and Puts

- **Put-call parity**
  - Put option value can be determined indirectly using Black-Scholes
  - For european options, on non-dividend paying stocks
    $C = P + S - ke^{-rt}$



Including Dividends in Black-Scholes

- **Two adjustment methods**

- **Assumption of constant dividend yield**
  - Replace S in formula with $s^*(1-d)^n$
    d = constant dividend yield
    n = number of dividend periods

- **Estimation of present value of dividends**
  - Replace S in formula with S-D
    D = present value of dividends

- **Put-call parity becomes either**

$$C = P + s^*(1-d)^n - ke^{-rt}$$

$$C = P + S - D - ke^{-rt}$$

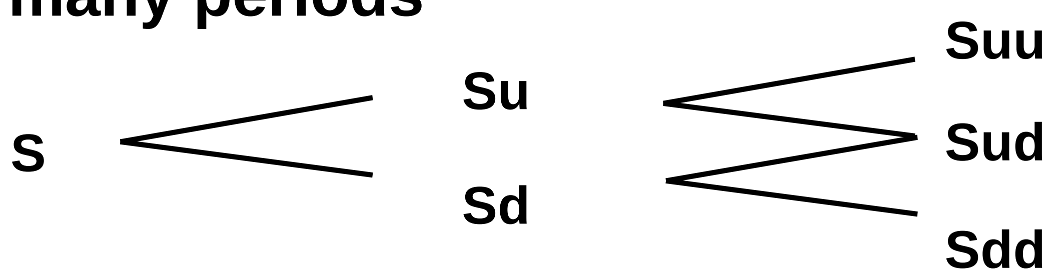
Limitations to Black-Scholes

- Black-Scholes values European Options
- Most Traded Options and most real options are American Type
- American Options can be exercised any time
  - In general, early exercise is not optimal (because option is more valuable than payoff)
  - Sometimes a valuable feature
- Overall, a more general approach is needed

A General Binomial Model for Options

- One-period call option example
  - Compared option value to portfolio of stock and borrowing
  - If stock price increased, call option had positive value
  - If stock price decreased, call option was worthless

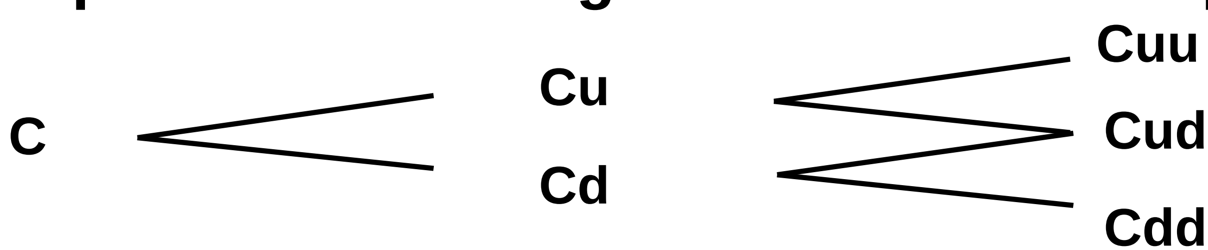
- In reality, stock price continues to change over many periods



u = up

d = down

- Option value changes each time stock price changes

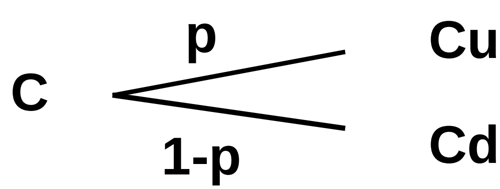


General Binomial Model Procedure

- Assumes many periods
- Works backward from date of expiration
- For each period, applies one-period valuation methodology
- At each node, compares
  - Value of option
  - Immediate exercise payoff
- Optimal policy determined for each period and stock price
  - Hold option for another period
  - Exercise immediately

General Binomial Model Results (Single Period)

- Value of call if held for single period



$$C = [p*cu + (1-p)*cd]/(1+r)$$

where, p acts as a probability

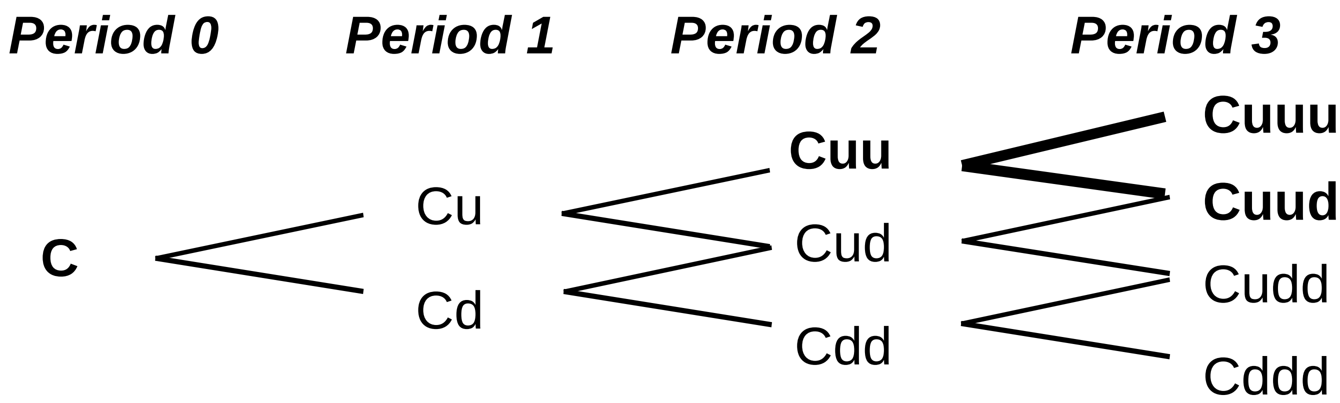
cu and cd determined by stock volatility

- Value of option is maximum of
  - Immediate exercise
  - Holding for another period
  - Zero

$$C = \max\{s-k, [p*cu + (1-p)*cd]/(1+r), 0\}$$

General Binomial Model Results (Multi-period)

- **Many periods are treated like a decision tree**



- **Work backward from last to first period to value C**

- **Apply one-period methodology at each node example:**
  cuu = max{suu-k, [p*cuuu + (1-p)*cuud]/(1+r), 0}

Comments on Binomial Model

- Binomial model is a recursive technique
  - Start with end-period values and work backward to present
  - Tedious for anything other than short examples
  - Can be automated in computer programs

- Note similarity to NPV
  - Estimate cash-flows (end-of-period option value)
  - Discount to present (using risk-free rate)

$$C = [p*cu + (1-p)*cd]/(1+r)$$

Crucial Role of Risk-free Discount Rate

- **Risk-free discount rate is used in options valuation**
- **Option valuation handles risk aversion by adjusting probability and discount rate**
  - Based on estimated cash-flows, an
  - Based on probability distribution of asset
  - The procedure adjusts their probability so that...
  - Risk-free rate is appropriate
  - No need to worry about what is appropriate risk-adjusted discount rate
- **The genius of the options valuation is precisely in the way this adjustment is done**

- **Options procedure is " risk neutral valuation"**
  - Critical concept of derivatives field

Summary of Valuation

- Value of options increases with
  - Value of asset, time available
  - Risk involved !!

- Options procedures use risk-neutral valuation
  - Adjust probabilities and cash flows so that risk-free rate can be used
  - Versus adjust discount rate and apply to cash-flows

- Black-Scholes is compact, but limited
  - Values European calls
  - Put-call parity works for valuing puts

- Binomial model more general
  - A recursive technique
  - More complicated, but can be automated

Appendix: observed option price influences

- **Combined List of Influences**

  Underlying Price (S)
  Strike Price (K)
  Time to Expiration (T)
  Risk-free Rate of Interest (R)
  Range (Volatility) of Stock Price Changes
  Dividends (D)
  American Vs European Options
  (Ability to Exercise Early)

Appendix: Impact of Individual Factors on Option Value

Factor/Option Type	American Call	American Put	European Call	European Put
Underlying Price	+	-	+	-
Strike Price	-	+	-	+
Time to Expiration	+	+	?	?
Volatility of Underlying	+	+	+	+
Risk-free rate of interest	+	-	+	-
Dividends	-	+	-	+

Appendix: Rationale for Influence Factors I

- Stock price
  - The greater the stock price (S) relative to strike price (K), the more likely a call (put) will be in (out of) the money
- Strike price
  - The greater the strike price (K) relative to stock price (S), the less likely a call (put) will be in (out of) the money
- Time to expiration
  - For American options, an option with a longer term to expiration is the same as an option with a shorter term, plus additional time
  - European options cannot be exercised until the expiration date, so the extra time could cause harm relative to the shorter term option

Appendix: Rationale Influence Factors II

- Volatility of underlying stock
  - Since options have a zero downside and a positive upside, increased volatility increases the likelihood of finishing in the money
- Risk-free rate
  - The strike price is paid or received in the future, and its present value is reduced by increased interest rates
  - For calls, the strike price is paid in the future
  - For puts, the strike price is received in the future
- Dividends:
  - Stock prices adjust downward for dividend payments. This reduces (increases) the likelihood a call (put) will finish in the money