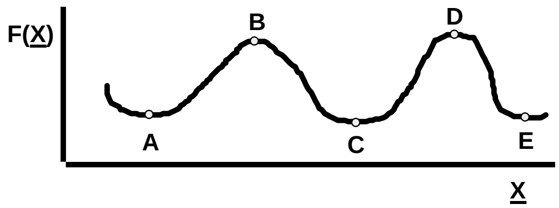
Constrained Optimization

- Unconstrained Optimization (Review)
- Constrained Optimization
  - Approach
  - Equality constraints
    - Lagrangeans
    - Shadow prices
  - Inequality constraints
    - Kuhn-Tucker conditions
    - Complementary slackness

Unconstrained Optimization (1)

- Definitions:
  - Optimization = Maximum of desired quantity
    = Minimum of undesired quantity
  - Objective Function = Expression to be optimized
    = $Z(\underline{X})$
  - Decision Variables = Variables about which we can make decisions
    = $\underline{X} = (X_1 \ldots . X_n)$

Unconstrained Optimization (2)



- **By calculus:**

If F(X) continuous, analytic:
Condition for maxima and minima

$$\partial F(X) / \partial X_i = 0 \qquad \forall_i$$

Unconstrained Optimization (3)

- **Secondary conditions:**

$\partial^2 F(\underline{X}) / \partial X_i^2 < 0$	= > Max	(B,D)
$\partial^2 F(\underline{X}) / \partial X_i^2 > 0$	= > Min	(A,C,E)

These define whether point of no change in Z is a maximum or a minimum

Unconstrained Optimization (4)

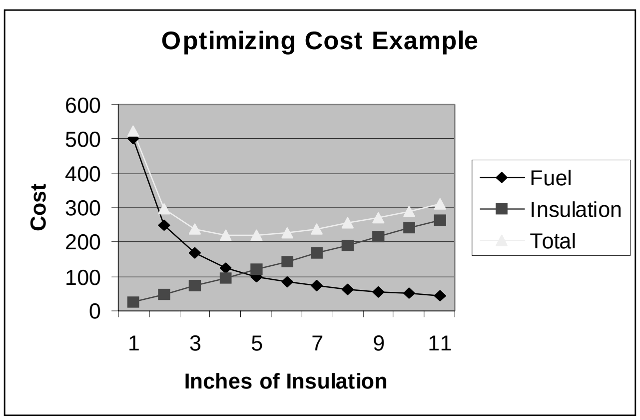
- Example: Housing insulation
  $F(x) = K_1 / x + K_2 x$
  Total Cost = Fuel cost + Insulation cost
  x = Thickness of insulation

  $\partial F(x) / \partial x = 0 = -K_1 / x^2 + K_2$

  $\Rightarrow x^* = \{K_1 / K_2\}^{1/2}$
  (starred quantities are optimal)

Unconstrained Optimization (5)

- $K_1 = 500$ $K_2 = 24$ $X^* = 4.56$

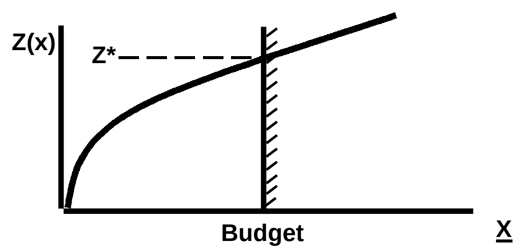


Constained Optimization

- "Constrained Optimization" involves the optimization of a process subject to constraints
- Constraints have two basic types
  - Equality Constraints -- some factors have to equal constraints
  - Inequality Constraints -- some factors have to be less less than or greater than the constraints (these are "upper" and "lower" bounds

Equality Constraints

- Example: Best use of budget
- Maximize: Output = $Z(\underline{X}) = a_o x_1^{a1} x_2^{a2}$
- Subject to (s.t.):
  Total costs = Budget = $p_1x_1 + p_2x_2$



Note: $\partial Z(\underline{X}) / \partial \underline{X} \neq 0$ at optimum

Constrained Optimization

- Approach

  To solve situations of increasing complexity, (for example, those with equality, inequality constraints) ...

  Transform more difficult situation into one we know how to deal with

- In this case, transform optimization of a "constrained" situation to optimization of "unconstrained" situation

Lagrangean Method (1)

- Transforms equality constraints into unconstrained problem
- Start with:

  Opt: F(x)

  s.t.: $g_j(\underline{x}) = b_j \Rightarrow g_j(\underline{x}) - b_j = 0$

- Get to:

  $L = F(\underline{x}) - \Sigma_j \lambda_j[g_j(\underline{x}) - b_j]$

  λj = Lagrangean multipliers (lambdas) -- these are unknown quantities for which we must solve

  Note: $[g_j(\underline{x}) - b_j] = 0$ by definition, thus

  optimum for $F(\underline{x})$ = optimum for L

Lagrangean Method (2)

- To optimize L:

  $\partial L / \partial x_i = 0 \qquad \forall_i$

  $\partial L / \partial \lambda_j = 0 \qquad \forall_i$

- Example:

  Opt: $F(\underline{x}) = 6x_1x_2$

  s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18$

  $L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18)$

  $\partial L / \partial x_1 = 6x_2 - 3\lambda = 0$

  $\partial L / \partial x_2 = 6x_1 - 4\lambda = 0$

  $\partial L / \partial \lambda_i = 3x_1 + 4x_2 -18 = 0$

Lagrangean Method (3)

- Solving as unconstrained problem:

  $\partial L / \partial x_1 = \quad 6x_2 - 3\lambda = 0$

  $\partial L / \partial x_2 = \quad 6x_1 - 4\lambda = 0$

  $\partial L / \partial \lambda_i = \quad 3x_1 + 4x_2 -18 = 0$

- so that: $\lambda = 2x_2 = 1.5x_1$

  $x_2 = 0.75x_1$

  $3x_1 + 3x_1 - 18 = 0$

- $x_1^* = 18/3 = 6 \quad x_2^* = 18/8 = 2.25 \quad \lambda^* = 4.5$
- $F(x)^* = 40.5$

Shadow Prices

- **Shadow Price is the Rate of change of objective function per unit change of constraint**

$$= \partial F(\underline{x}) / \partial b_j$$

- **This is meaning of Lagrangean multiplier**

  $SP_j = \partial F(\underline{x})^* / \partial b_j = \lambda_j$

  – **Naturally, this is an instantaneous rate**
- **The shadow price is extremely important for system design**
- **It defines value of changing constraints**

Shadow Prices (2)

- **Let's see how this works in example, by changing constraint by 0.1 units:**

  Opt: $F(\underline{x}) = 6x_1x_2$

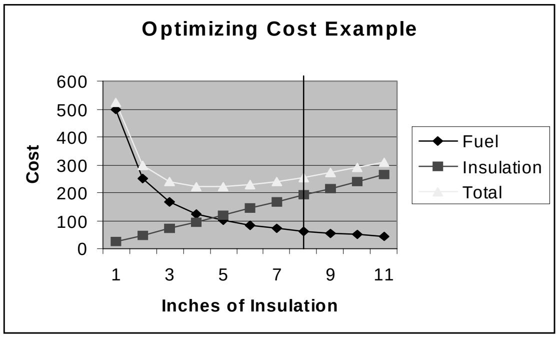
  s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18.1$
- **The optimum values of the variables are**

  $x_1^* = (18.1)/6$ $\quad$ $x_2^* = (18.1)/8$
- **Thus** $F(x)^* = 6(18.1/6)(18.1/8) = 40.95$

  $$\Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1)$$

Inequality Constraints

- Example: Housing insulation
  Min: Costs = $K_1 / x + K_2x$
  s.t.: $x \geq 8$ (minimum thickness)



Inequality Constraints (2)

- Approach: Transform inequalities into equalities, then proceed as before
- Again, introduce new variable -- the "Slack" variable that defines "slack" or distance between constraint and amount used
- The resulting equations are known as the "Kuhn-Tucker conditions"

Inequality Constraints -- insertion of slack variables in Lagrangean

- A "slack variable", s_j , for each inequality
  $g_j(\underline{x}) \le b_j \quad => \quad g_j(\underline{x}) + s_j^2 = b_j$
  $g_j(\underline{x}) \ge b_j \quad => \quad g_j(\underline{x}) - s_j^2 = b_j$
- These are "squared" to be positive
- start from:
  opt: $F(\underline{x})$ s.t.: $g_j(\underline{x}) \le b_j$
- get to:
  $L = F(\underline{x}) - \Sigma_j \lambda_j [g_j(\underline{x}) + s_j^2 - b_j]$

Inequality Constraints -- Complementary Slackness Conditions

- The optimality conditions are:
  $\partial L / \partial x_i = 0$
  $\partial L / \partial \lambda_j = 0$
  plus: $\partial L / \partial s_j = 2 s_j \lambda_j = 0$
- These new equations imply:
  $s_j = 0 \qquad \lambda_j \neq 0$
  or
  $s_j \neq 0 \qquad \lambda_j = 0$

  They are the "complementary slackness" conditions. Either slack or lambda =0 $\forall_i$

Interpretation of Complementary Slackness Conditions

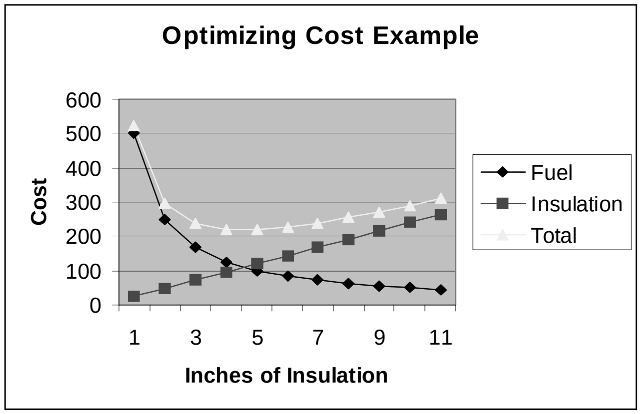
- **Interpretation:**
- **If there is slack on b_j, (i.e. more than enough of it)**
  => No value to objective function to having more: $\lambda_j = \partial F(\underline{x}) / \partial b_j = 0$
- **If $\lambda_j \neq 0$, then all available b_j used**
  => $s_j = 0$

Application to Example

- **Min: Costs = $K_1 / x + K_2 x$**
  s.t.: $x \geq 8$ (minimum thickness)
- **$L = K_1/x + K_2 x - \lambda[x - s^2 - b]$**
- **$L = 500/x + 24x - \lambda[x - s^2 - 8]$**
- **$500/x^2 + 24 - \lambda = 0$ $2\lambda s = 0$ $x - s^2 = 8$**
- **If $s = 0$, $x = 8$, $\lambda = 31.8$ (at that point)**
  Max = 254.5
- **Therefore, worth relaxing (in this case, lowering) constraint to get maximum**
- **$x^* = 4.56$ Optimum = 221**

Unconstrained Optimization (5)

- $K_1 = 500$ $K_2 = 24$ $X^* = 4.56$



Another application to Example

- Min: Costs = $K_1 / x + K_2 x$
  s.t.: $x \geq 4$ (NEW MINIMUM)
- $L = K_1/x + K_2x - \lambda[x + s^2 - b]$
- $L = 500/x + 24x - \lambda[x + s^2 - 4]$
- $500/x^2 + 24 - \lambda = 0$ $2\lambda s = 0$ $x - s^2 = 4$
- If $\lambda = 0$, $x = 4.56$, slack, $s^2 = 1.56$
  Optimum = 221
- not worth changing constraint

Summary of Presentation

- Important mathematical approaches
  - Lagreangeans
  - Kuhn- Tucker Conditions
- Important Concept: Shadow Prices
- THESE ANALYSES GUIDE DESIGNERS TO CHALLENGE CONSTRAINTS