Production Functions

Outline

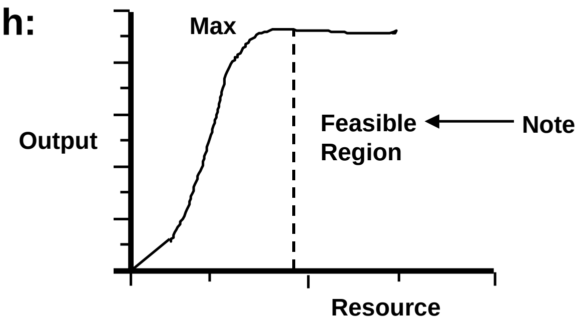
1. Definition
2. Technical Efficiency
3. Mathematical Representation
4. Characteristics

Production Function - Basic Model for Modeling Engineering Systems

- **Definition:**
  - Represents technically efficient transform of physical resources $X = (X_1 \ldots X_n)$ into product or outputs Y (may be good or bad)
- **Example:**
  - Use of aircraft, pilots, fuel (the X factors) to carry cargo, passengers and create pollution (the Y)
- **Typical focus on 1-dimensional output**

Technical Efficiency

- **A Process is Technically Efficient if it provides Maximum product from a given set of resources $X = X_1 , ... X_n$**
- **Graph:**



Mathematical Representation - General

- **Two Possibilities**
- **Deductive -- Economic**
  - Standard economic analysis
  - Fit data to convenient equation
  - Advantage - ease of use
  - Disadvantage - poor accuracy
- **Inductive -- Engineering**
  - Create system model from knowledge of details
  - Advantage - accuracy
  - Disadvantage - careful technical analysis needed

contrast

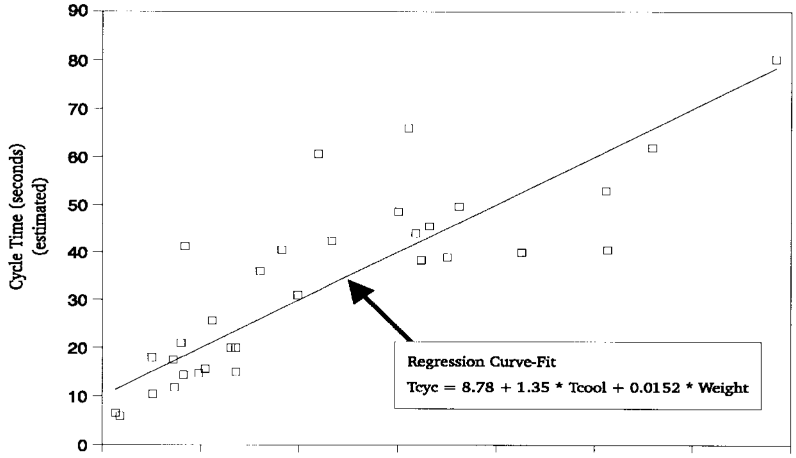
Mathematical Representation - Deductive

- Standard Cobb-Douglas Production Fnc.
  - $Y = a_0 \pi X_i^{a_i} = a_0 X_i^{a_i} \ldots X_n^{a_n}$
  - Interpretation: 'a_i' are physically significant
  - Easy estimation by linear least squares
    $\log Y = a_0 + \Sigma a_i \log X_i$
- Translog PF -- more recent, less common
  - $\log Y = a_0 + \Sigma a_i \log X_i + \Sigma\Sigma a_{ij} \log X_i \log X_j$
  - Allows for interactive effects
  - More subtle, more realistic

Mathematical Representation - Inductive

- "Engineering models" of PF
- Analytic expressions
  - Rarely applicable: manufacturing is inherently discontinuous
  - Exceptions: process exists in force field, for example transport in fluid, river
- Detailed simulation, Technical Cost Model
  - Generally applicable
  - Requires research, data, effort
  - Wave of future -- not yet standard practice

Cooling Time, Part Weight, and Cycle Time Correlation



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PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Convexity of Feasible Region

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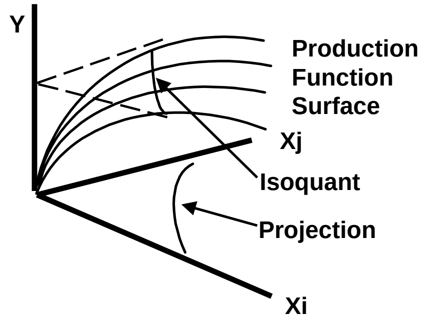
Characteristic: Isoquants

- Isoquant is the Locus (contour) of equal product on production function
- Graph:



Important Implication of Isoquants

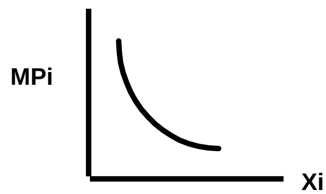
- Many designs are technically efficient
  - All points on isoquant are technically efficient
  - no technical basis for choice among them
  - Example:
    - little land, much steel => tall building
    - more land, less steel => low building
- System Design depends on Economics
- Values are decisive

Characteristic: Marginal Products

- Marginal Product is the change in output as only one resource changes

$$MP_i = \partial Y / \partial X_i$$

- Graph:



Diminishing Marginal Products

- Math:

$$Y = a_0 X_1^{a_1} \ldots X_i^{a_i} \ldots X_n^{a_n}$$

$$\partial Y / \partial Xi = (ai/Xi)Y \quad = f \; (Xi^{a_i - 1})$$

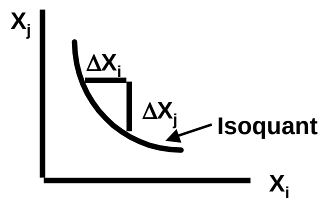
Diminishing Marginal Product if $a_i < 1.0$

- "Law" of Diminishing Marginal Products
  - Commonly observed -- but not necessary
  - "Critical Mass" phenomenon => increasing marginal products

Characteristic: Marginal Rate of Substitution

- Marginal Rate of Substitution is theRate at which one resource must substitute for another so that product is constant

- Graph:



Marginal Rate of Substitution (cont'd)

- Math:

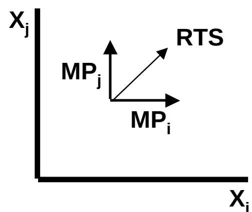
  since $\Delta X_i MP_i + \Delta X_j MP_j = 0$
  (no change in product)

  then $MRS_{ij} = \Delta X_i / \Delta X$
  $= - MP_j/MP_i = - (a_j/a_i)(X_i/X_j)$

- MRS is "slope" of isoquant
  - Note: It is negative
  - Loss in 1 dimension made up by gain in other

Characteristic: Returns to Scale

- Returns to Scale is the Ratio of rate of change in Y to rate of change in ALL X (each X_i changes by same factor)
- Graph:
  - Directions in which the rate of change in output is measured for MP and RTS



Returns to Scale (cont'd)

- Math:

$Y' = a_0 \pi X_i^{a_i}$

$Y'' = a_0 \pi (sX_i)^{a_i}$

$= Y'(s)^{\Sigma a_i}$

$RTS = (Y''/Y')/s = s^{(\Sigma a_i - 1)}$

Y''/Y' = % increase in Y

if $Y''/Y' > s$ => Increasing RTS

Increasing returns to scale if $\Sigma a_i > 1.0$

Importance of Increasing Returns to Scale

- **Increasing RTS means that bigger units are more productive than small ones**

- **IRTS => concentration of production into larger units**

- **Examples:**
  - **Generation of Electric power**
  - **Chemical, pharmaceutical processes**

Practical Occurrence of Increasing Returns to Scale

- **Frequent!**
- **Generally where**
  - * **Product = *f* (volume) and**
  - * **Resources = *f* (surface)**
- **Example:**
  - * **ships, aircraft, rockets**
  - * **pipelines, cables**
  - * **chemical plants**
  - * **etc.**

Characteristic: Convexity of Feasible Region

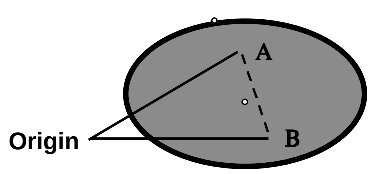
- A region is convex if it has no “reentrant” corners

- Graph:

Test for Convexity of Feasible Region (cont’d)

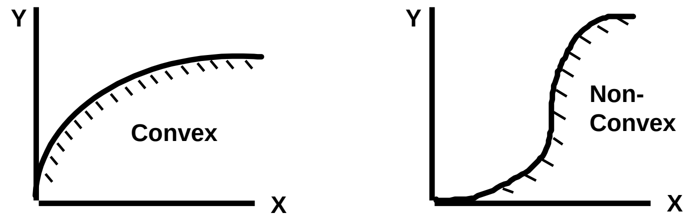
- Math: If A, B are two vectors to any 2 points in region

  Convex if all
  $T = KA + (1-K)B \qquad 0 \leq K \leq 1$
  entirely in region



Convexity of Feasible Region for Production Function

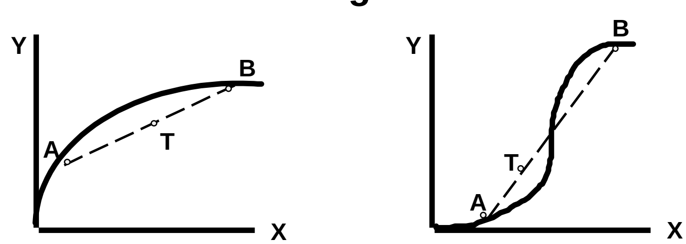
- Feasible region of Production function is convex if no reentrant corners



- Convexity => Easier Optimization
  - by linear programming (discussed later)

Test for Convexity of Feasible Region of Production Function

- Test for Convexity: Given A,B on PF
  If T = KA + (1-K)B $0 \leq K \leq 1$
  Convex if all T in region



- Cobb-Douglas: $a_i \leq 1.0$ and $\Sigma a_i \leq 1.0$