APPLIED SYSTEMS ANALYSIS

Engineering Planning and Technology Management

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McGraw-Hill Publishing Company
New York St. Louis San Francisco Auckland Bogotá Caracas Hamburg
Lisbon London Madrid Mexico Milan Montreal New Delhi Oklahoma City
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18.4. *Threshhold*

(*a*) As the founder of a new engineering company you must earn $10,000 a month to pay rent, salaries, and the bank loan. Anything over this is your own salary. Anything under must be added to your debt. Think about and plot the form of your utility for a monthly income of $5000 to $20,000.

(*b*) As an employee in a big company, you know that the average salary increase for your colleagues is 5%. Variations depend on your bosses' assessment of your performance. Plot the form of your utility for a salary increase of 0 to 15%.

18.5. *Strategic Equivalence*

The sales, S, of a new high-tech product are estimated to depend on price, P (measured in $\$ \times 10^3$) and quality C (measured by an index):

$$S = 300P^{-0.5}C$$

(*a*) Plot price versus quantity (the demand function) for $C = 1, 2, 3$. Interpret the meaning of these curves.

(*b*) Suppose we introduce a new index of quality: $C' = 3C^2$. How would we change the formula for sales? How would the new index affect the analysis?

18.6. *Scales*

(*a*) What are the positive linear transformations that give °C as a function of °F? K as a function of °C?

(*b*) Suppose a utility function for X is a linear interpolation between the following points:

X	:	$0	$20	$50	$100
$U(X)$	:	0	0.1	0.8	1.0

Plot $U(X)$. Then plot $U'(X)$ as the positive linear transformation with a range of 0 to 0.4; 0.8 to 1.0; 0.5 to 0.8.

(*c*) The utility for response time, T, to an emergency is

T (min)	:	0	5	10	15	20
$U(T)$	:	1.0	0.8	0.5	0.1	0

Plot $U'(T)$ as the positive linear transformation with a range of 40 to 80.

(*d*) The utility for deaths, D, due to industrial accidents is

D(units)	:	0	1	2	5	10
$U(D)$	:	0	−0.01	−0.05	−0.30	−1.0

Plot $U'(D)$ as the positive linear transformation with a range of 0 to 100.

CHAPTER 19

MEASUREMENT OF UTILITY

19.1 ORGANIZATION

This chapter presents the basic procedures for measuring utility: the Certainty Equivalent (CE) and the Lottery Equivalent/Probability (LEP) methods. The CE approach has been the standard and is still widely used in practice. It suffers from a broad range of practical defects, however. This has led to the development of the LEP, which experts in the measurement of utility now recommend.

Many other procedures are possible. As the next section indicates, we measure utility by solving a simple linear equation with one unknown and a minimum of three independent arguments. These can be permuted in many ways, leading to a variety of measurement procedures, each emphasizing different features. Research into these possibilities is active, and we can expect additional improvements in utility measurement.

The techniques for measuring utility blend two contrasting intellectual paths. First there is the theory of utility, presented in Chapter 18. Then there is *psychometrics*, the science of measuring behavioral responses of people (or animals). Psychometrics has been developed and applied extensively in behavioral research for nearly a century. It demonstrates the techniques that should be used to obtain good measurements of utility, those that are replicable with small margins of error or variance.

The proper presentation of utility measurement is thus inherently complex because we must both integrate psychometrics and cover alternative methods. To simplify matters, we have adopted the following organization. Sections 19.2 and 19.3 provide the theoretical and psychometric basis. Sections 19.4 and 19.5 describe the CE and LEP methods and place them in the overall step-by-step

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procedure. Sections 19.6 and 19.7 cover special situations. Sections 19.8 and 19.9 illustrate the measurement itself and the application of utility to decision analysis.

19.2 GENERAL CONDITIONS

Measurement of utility is based on the central result of the axioms of utility:

$$U(\text{Lottery}) = \sum P_i U(\mathbf{X}_i)$$

The utility of a risky situation is the sum of the utility of each of the possible outcomes times their probability of occurrence. This provides us with a simple linear expression that we can solve explicitly for the utility of any specific level of X.

The methods for determining utility have a common characteristic: they establish an equivalence between a stimulus and a response. The stimulus is provided in the measuring process to provoke a person's response, indicating the intensity of preference, the utility. The process is analogous to radar identification of objects: a signal goes out and the response interpreted.

The stimulus will be, in practice, a binary lottery of the form $(X_1, P;\ X_2)$. It will thus have three independent parameters since the probability of X_2 must be complementary to that of X_1 (see Section 18.5). The stimulus is structured so that its utility is known, either by calculation or construction as described in Section 19.4.

The response provided by a person is designed to define a situation of equal value to the stimulus. The key factor is that it contains no more than one $\mathbf{X}_i$ whose utility is unknown:

Utility of Response (with one Unknown Utility $\mathbf{X}_i$)

= Utility of Stimulus (of Known Utility)

Solution of this equation provides the utility of $\mathbf{X}_i$. In principle, the procedure can be continued indefinitely to obtain as many readings on $U(\mathbf{X})$ as desired. The limits on the process are practical.

There are many ways to obtain the utility function. These are associated with the many ways one can define the stimulus and the response. The sequence of observations can be obtained by keeping the probabilities in the stimulus lottery constant and varying the outcomes, or by varying the probability while keeping the outcomes constant. The size and sequence of the gradations in the outcomes or the probabilities can be taken in any convenient order. Finally, the response can be either a deterministic quantity (a *certainty equivalent*) or another lottery (a *lottery equivalent*).

Measuring utility is a process similar to triangulation used in surveying. Here one seeks to define an unknown utility function over $\mathbf{X}$ by some chain of observations based on the assumed utility of two values of $\mathbf{X}_i$ that define the scale for utility. Triangulation similarly attempts to define terrain, a shoreline for example, based on bearings taken from two positions whose location is assumed to be known.

Theoretically, if the axioms are correct, all these procedures should lead to the same utility function for an individual, constant to a positive linear transformation. If the base pair of outcomes is assigned the same arbitrary values, the utility functions should be identical. It is the same with triangulation: no matter which way you go about it, you should end up with the same trace for the shoreline. The difficulty, as we will see in the discussion of the standard certainty equivalent method in Section 19.4, is that the theory sometimes fails. These situations have to be recognized and avoided.

As a practical matter, the procedures for measuring utility functions apply to situations in which the outcomes have only one dimension, that of money, for instance. The subsequent discussion in this chapter therefore focuses on $U(X)$. The procedures for assessing multiattribute utility functions, $U(\mathbf{X})$, in which the outcomes have several dimensions (money, safety, pollution, etc.), are presented in Chapter 20. The multiattribute utility functions are usually constructed from one-dimensional utility functions.

Since both value and utility functions are unique to individuals, as indicated in Section 18.3, the measurement of any utility function must be done in some kind of personal way. This can be through individual interviews or by questionnaires administered collectively. Because the methods are so personal, doing them well requires some special psychometric skills.

19.3 PSYCHOMETRIC CONSIDERATIONS

At least five issues need to be considered in the measurement of human response. These concern:

1. The nature of the interview
2. Its context
3. The scale of the response
4. The way it is obtained
5. Consistency and replicability

Interview. The measurement of a person's utility function necessarily requires some sort of personal interaction with this individual. Measurement of personal choice and values is quite different from ordinary engineering measurements of materials and things. People tend to respond to their interviewer on a personal basis, as materials evidently do not. A person feeling tested or threatened will normally distort his or her truthful responses to socially acceptable ones, suitable to the occasion. For example, if the tax service asks people about their income, they normally tend to minimize it. Care must be taken to avoid biases that this kind of behavior introduces into the measurements.

A basic rule for the measurement process is thus to put the person being interviewed at ease with the process. It is important to stress that

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1. The analyst requires and values the interviewees' opinion. They are the only ones who can provide it, and are thus the experts.
2. There are no wrong answers (the interviewees are entitled to their preferences, even when these do not accord with the axioms—the simplifying assumptions—that are the basis of the theory).

The second point is particularly important because people being interviewed by analysts tend to feel that they are being tested to see if they are correct or as smart as the analyst. This may lead them to distort their answers toward what they think the analyst wants to hear. It certainly may put them on the defensive and hinder the measurement process.

Context. It is important to recognize that people's utility functions depend on their situation. For example, your valuation of food is different when you are starving or have just eaten. Similarly, people's valuation of money clearly depends on their current financial position. After you have just been paid, or received a generous holiday gift, you might feel more inclined to take a financial risk than if you were trying to make ends meet at the end of the month. The utility function for losses is also generally different in intensity and curvature from the utility function for gains.

The extent to which utility functions depend on context is a matter of current research. Meanwhile, it is good practice to measure a person's utility in a context that is relevant to them and the decisions they may take.

One way to place the questions in a relevant context is to suggest professionally or personally familiar situations to the interviewee. Another is to present scenarios similar to the projects or strategies that will be evaluated with the person's utility function. It is best if both these variants can be merged.

The questionnaires used in a recent evaluation of ways to mitigate against damage to buildings from earthquakes illustrate the context issue. In this case, the ultimate question was how much expensive reinforcement should be put in buildings to protect the public. The optimal choice naturally depended on whether you were a tenant and had to pay higher rent; a designer, interested in making sure your building never collapsed; and so on. To define the optimal design from the perspective of different groups, we had to obtain their utilities for money. To get these, we recognized their different backgrounds, and prepared questionnaires that asked the same mathematical questions, but in a customary context for members of each group (see box).

Scale. The scale of the utility function is defined by means of two arbitrary points. This is as for any ordered metric scale such as temperature (see Section 18.6). Since the points are arbitrary, they might as well be convenient.

Practitioners usually find that the best points to define the scale are those at the extremes of the range of X with which they are concerned. Conventionally, the worst level of X is labelled X_*, and the best X^*. It is typically most convenient to assign these the utility values of

Placing Questions in Different Contexts

In our study to determine the best level of protection against earthquakes, we wanted to find people's utility for extra costs. To make them comparable, these were expressed as percentage increases over normal costs, from 0% to +10%. Our stimulus was thus to be a lottery of the form:

$$(0\% \text{ extra}, P = 1/2; \quad 10\% \text{ extra})$$

This situation was presented in an appropriate context for each group. For engineering contractors we stated:

> "Suppose you are in charge of the construction of some building. Early results on the tests of concrete from the foundation indicate that it is understrength. On the best expert advice, you perceive that you have two choices, either to rip out the apparently weak spot (Choice A) or to leave it in. If you leave it in, you may never have to touch the foundation (Outcome B) or you may have eventually to go back, rip out some completed work, and pay a lot (Outcome C). You estimate that B and C are equally likely to occur if you leave the suspicious concrete in. The cost of B is 0% extra on the total cost of the building, and the cost of C is a 10% increase."

To tenants we interviewed we said, however:

> "Suppose you are forced to give up your sublet and look for another apartment. As you hunt around, you see you have really two choices, either to take what is immediately available elsewhere in your apartment building (Choice A) or to move. If you move you may find something at your current rental (Outcome B) or may have to pay considerably more (Outcome C). You estimate that B and C are equally likely to occur if you decide to move. The cost of B is 0% more than your current rental, and the cost of C is a 10% increase in rent."

In both cases the person being interviewed faced the same choice:

Choice A versus The stimulus

$$U(X_*) = 0 \qquad U(X^*) = 1.0$$

If X is some undesirable outcome, for example the number of accidental deaths, X_* might not be the lowest amount. This implies a utility function decreasing to the right. If this seems bothersome one can simply redefine the axes, for example by making X a desirable quantity such as accidental deaths prevented. If one does not like the idea of assigning a positive utility to undesirable outcomes, one can also run the utility scale from -1.0 to 0 so that

$$U(X_*) = -1.0 \qquad U(X^*) = 0$$

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Bracketing. A special procedure should be used to obtain equivalents from people: bracketing. This is a way to help people define equivalents, a task which many ordinarily find difficult. The essential idea is for the interviewer to help the person focus on the equivalence gradually. The interviewer does this by first suggesting an equivalent that is less than the maximum but almost certainly too high, and to ask if the person being interviewed prefers the equivalent or the test lottery. The answer is easy. The interviewer next suggests an answer that is probably too low. Again an easy answer. The result so far is to narrow the possible range of response by "bracketing" the equivalent which lies somewhere in between. The interviewer continues the process until the interviewee settles on the equivalent. This is illustrated by the example from the earthquake questionnaire (see box).

Bracketing works in two ways. One is to focus on the answer gradually, helping the respondent exclude the easy cases and concentrate on the answer. The other is to reach the answer both from above and below. This avoids the phenomenon known as *anchoring*, the tendency people have to give a high answer when it is approached from above, and a low answer when approached from below. Good psychometric practice mitigates this bias by taking measurements from both directions. Tests of hearing are generally conducted in this way. Bracketing achieves the same result.

Computer program. An experienced interviewer can avoid introducing biases into the measurement of utility, can be consistent in the way of phrasing questions and presenting choices, and can thus obtain reliable results with low error. In general this requires skills not ordinarily available. Many measurements of utility thus contain mistakes made by the interviewers.

Interactive computer programs provide the means to avoid these errors and insure consistency and reliability in the measurement of utility. Properly done, they incorporate the significant skills of an experienced interviewer, and are thus expert systems in some sense. They also have many additional features that improve the measurement. They are, obviously, totally consistent and cannot bias the responses by the different ways a person might present the stimuli. People being interviewed also tend to feel more comfortable working with a machine than with a person, whom they suspect may be judging them.

Computer programs for utility assessment provide great analytical advantages over oral or written interviews. They encode the data instantaneously, avoiding the tedious task of transcribing responses. They also can process it immediately to provide feedback as appropriate, for example with plots of the utility function. Overall, research has demonstrated that interactive computer programs for measuring utility reduce the band of error, the variance, significantly. One example of an interactive computer program for measuring utility is available for educational and research purposes from the author at MIT. This ASSESS program is written in Fortran and C.

Application of Bracketing

The questionnaires used in the earthquake study used bracketing to get the equivalence of the response and the stimulus. The following kinds of questions came after the situations presented in the previous box:

Would you choose A if it cost	1% extra?	yes	no
	8% extra?	yes	no
	3% extra?	yes	no
	7% extra?	yes	no
	5% extra?	yes	no
	6% extra?	yes	no

What is the maximum you would be prepared to pay for Choice A before you would be prepared to take the risk?

The respondent would, by answering each question, be closing in on the value appropriate for this person.

19.4 ALTERNATIVE RESPONSES

This section discusses two kinds of responses that can be solicited to obtain utility functions: the certainty equivalent and the lottery equivalent/probability. The certainty equivalent has been the standard and must therefore be known. Its numerous defects have motivated the search for a replacement. The lottery equivalent/probability is the recommended alternative.

Certainty equivalent (CE). The conventionally standard method for measuring utility uses certainty equivalents. To start, the stimulus is the binary lottery whose possible outcomes are the extremes of the range, as defined in the previous section. The first point on the utility function is defined by the certain amount that a person values equally to the stimulus. This quantity is the *certainty equivalent*, X_1 :

$$X_1 \sim (X^*, P; \quad X_*)$$

Given our definitions, $U(X_1) = P$, as indicated in Figure 19.1.

The next two points on the utility function are obtained by substituting the certainty equivalent X_1 for each of the extreme levels of X in the binary lottery and obtaining new certainty equivalents, X_2 and X_3. Thus,

$$X_2 \sim (X_1, P; \quad X_*)$$

$$X_3 \sim (X^*, P; \quad X_1)$$

giving $U(X_2) = P^2$ and $U(X_3) = P(2 - P)$. Subsequent points are obtained by substituting X_2 and X_3 in the binary lottery for X^*, X_1, and X_* in turn. The process can be repeated as often as desirable or practical.

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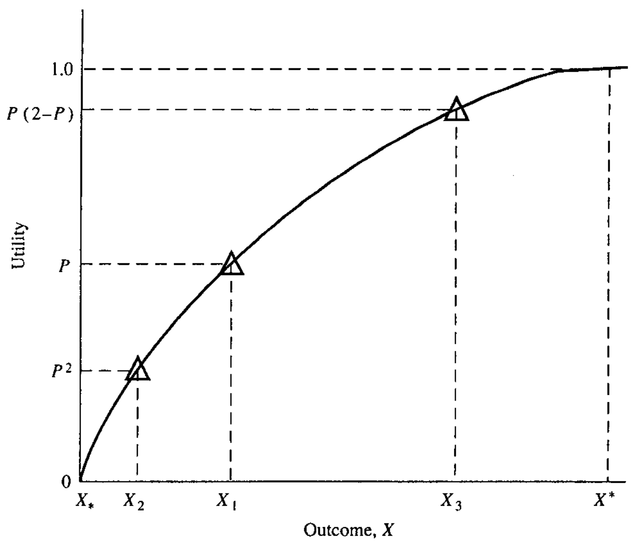

FIGURE 19.1
Measurement of the utility function $U(X)$ by the fractile method.

This approach, using the same probability in all stimulus lotteries, is also called the *constant probability* method. This is in distinction to the *variable probability* method, which seeks the certainty equivalents to binary lotteries in which the interviewer provides different probabilities as a stimulus. The latter method is really only used in research.

The procedure is particularly simple if $P = 0.50$. Then the certainty equivalents divide the range of utility into halves, quarters, and so on. Indeed, $U(X_1) = P \quad U(X^*) = 0.5$, and then $U(X_2) = P \quad U(X_1) = 0.25$ as well as $U(X_3) = P \quad U(X^*) + (1 - P) \quad U(X_1) = 0.75$. For this reason the approach is also called the *fractile method*. This division into equal parts has a particular advantage: it makes it possible to check the measurement of the utility function for internal consistency. Since $U(X_1)$ is halfway in value between $U(X_2)$ and $U(X_3)$ by construction, consistent measurements should find that the certainty equivalent to $(X_3, 0.5; \quad X_2)$ is X_1. This can be verified by asking whether

$$(X_3, 0.5; \quad X_2) \sim X_4 = X_1?$$

With a little practice, analysts find it easy to obtain consistent responses (±5%) for measurements.

Most people also are more ready to respond to questions put in terms of 50:50 lotteries. They are usually familiar with the idea of bets on the toss of a coin, as in "heads or tails?" This certainly makes the measurement easier. It does not, however, necessarily make it better. It is possible that people get into a rut about 50:50 lotteries and give distorted responses. This is an open question for research.

Most unfortunately, the certainty equivalent method of assessing utility has numerous, major defects. Extensive recent experiments indicate that, for reasons that are not yet clear, measurements made with this method appear to be systematically distorted.

The empirical fact is that measurements of the utility function using different probabilities in the fractile method translate into quite different results. All applications of the fractile method should by theory lead to the same utility function when applied to an individual, just as triangulation should trace the same shoreline regardless of the angles of the lines of sight. The problem is that they do not: utility functions obtained using higher probabilities in the stimulus lottery are systematically above the utility functions obtained with lower probabilities (see Figure 19.2). This family of curves is easily different by ±25% on the ordinates. The reason for this difficulty has not been determined, but a certainty effect has been suspected (See Section 18.5 and box on Allais paradox).

Additionally, the standard certainty equivalent method has other defects that encourage and propagate errors in measurement. By construction, for example, the stimulus constantly changes the range of choices presented to a person; it varies from (X^* to X_*) to (X^* to X_1) to (X_1 to X_*), and so on. This is confusing, and is likely to induce errors. Compounding this effect, the process inherently propagates errors since each reading depends on previous results. This chain of measurement passes any error on to all subsequent measurements. This is contrary to good practice, which would insist on independent readings for each value.

These theoretical and practical difficulties with the certainty equivalent method have motivated the development of alternative procedures. The most likely candidate as a replacement now appears to be the lottery equivalent/probability method.

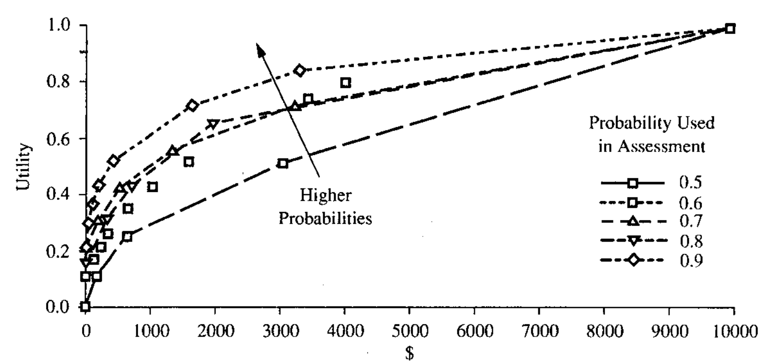

FIGURE 19.2
Empirical demonstration that utility functions obtained by the fractile method are layered according to the probabilities used in the assessment.

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Lottery Equivalent/Probability (LEP). This new procedure uses lottery equivalents instead of certainty equivalents. It thereby simultaneously avoids difficulties associated with the certainty effect and the chaining of error.

The lottery equivalent method uses a binary lottery for the response. In the recommended version, the outcomes are fixed at the extremes of the range of X, and the probability varies in each measurement. Formally, it uses $(X^*, P_e;\quad X_*)$. This lottery is made equivalent to the binary lottery by adjusting the probability P_e, which is the response in the LEP method.

In practice the LEP method works as follows. The binary lottery with P_e is compared to another binary lottery with $P = 0.50$ and one outcome set at the worst end of the range of X, X_*, and the other at the variable X_i. Formally,

$$(X^*, P_e;\quad X_*) \sim (X_i, 0.50;\quad X_*)$$

Successive (X_i, P_e) pairs define the utility function.

The arithmetic of the LEP method is direct. With the conventional assumptions that $U(X^*) = 1.0$, $U(X_*) = 0$, each lottery equivalent leads to

$$U(X_i) = 2P_e$$

Figure 19.3 illustrates the procedure.

The obvious advantage of this approach is that it avoids all reference to certainty and thereby absolutely excludes the possibility of certainty effects. A

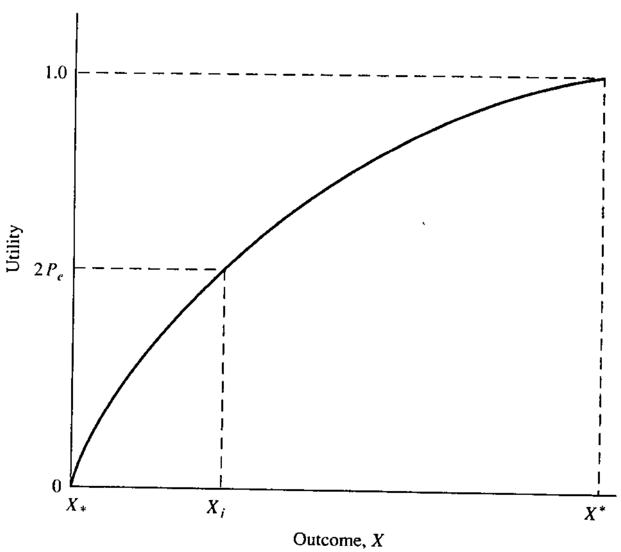

FIGURE 19.3
Measurement of the utility function $U(X)$ by probability equivalents.

secondary advantage is that it avoids error propagation: each measurement of the utility function is independent of others.

Experiments to date with the lottery equivalent/probability method indicate that it provides a much tighter set of measurements of the utility function than the fractile method. The results are, however, not yet conclusive.

There is another lottery equivalent method, the lottery equivalent/outcome approach. This fixes the probability of the response and varies one of the outcomes. Research indicates that this method leads to a family of utility functions that depend on the probability, rather as the certainty equivalent method does. The lottery equivalent/outcome method is thus so far only used in research.

19.5 STEP-BY-STEP PROCEDURE

The practical process of measuring utility consists of five distinct steps:

1. Defining X
2. Setting Context
3. Assessment
4. Interpretation
5. Functional Approximation

1. Defining X. To start the process we must first define what it is we want to measure: the nature of X, its range, and thus its scale. Because utility functions depend on their context, as indicated in Section 19.3, we must therefore determine how we will be using the utility functions and in which decision analyses they will be used.

Normally, the choice of X and its range follow directly from the problems in which the utility function will be used. The nature of X will be determined by the way the decision problem has been formulated. The extremes of the range, X^* and X_*, will simply be the extreme outcomes of the problem.

2. Setting context. We must next develop a way to present the interview to the person whose utility we are measuring. Specifically, we must prepare a scenario that puts risky choices over X in a context appropriate to the decision analysis and to the person.

For example, consider the problem of choosing the right level of earthquake protection for buildings, as sketched out in Section 19.3. In this instance the decision analysis concerned costs and utility functions for money were required. The problem itself was about housing. Merging these two situations, we developed a context that referred to housing costs. This concept was further adjusted to the people being interviewed: design engineers were asked about costs in the construction of buildings; consumers were asked about rental costs for apartments.

3. Assessment. The assessment itself proceeds as described in the previous Section. This is illustrated in detail, for both the certainty equivalent and lottery equivalent/probability methods, in Section 19.8.

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4. Interpretation. The pairs of equivalences between stimulus and responses must be solved for values of $U(X_i)$. This is done as indicated previously. Normally, $U(X)$ is then plotted by interpolation between the measured points.

5. Functional approximation. This step is optional. It consists of fitting an analytic function to the measurements so that the utility function can be used more easily in the decision analysis. Two kinds of functional forms are usual.

The exponentional function has been favored in the past. It is

$$U(X) = a + be^{-cX}$$

This function has some nice properties mainly to do with the obsolescent concept of "risk aversion." These are irrelevant for practical problems.

The alternative is the power function

$$U(X) = a + bX^c$$

In practice it appears to be as effective as the exponential. Either is fitted to the data in the same way: by adjusting the functional parameters so that the function generates the values of $U(X)$ assumed at the endpoints of the range and gives the best fit of intermediate points.

19.6 BUYING AND SELLING LOTTERIES

The interviewer should be careful about the way the stimulus lottery is presented and interpreted. The issue centers on whether the person being interviewed sees the situation as "buying" or "selling" the lottery. These conventional terms generally cause problems because the interviewer is trying to determine whether a person is indifferent between two quantities, not to buy or sell anything. The issue is nevertheless real: it concerns the interpretation of the X. It thus directly affects the assessment of the utility function.

The problem is best explained by example. Consider a lottery ticket for a prize, $(X^*, P;\quad 0)$, and its equivalents. Keep the arithmetic simple; consider only certainty equivalents, although the issue is general.

Selling the lottery. One possibility is that you have the ticket. You can then exchange or sell it for a certain amount (presumably somewhere in between X^* and 0). Your *selling price*, X_s, is the price at which you are indifferent between keeping the risk and selling it; it is the certainty equivalent for the lottery: $X_s \sim (X^*, P;\quad 0)$. Note that in this case your three possible outcomes are

$$0, \quad X_s, \quad X^*$$

You either walk away with the certain amount, X_s, or end up with or without the prize (X^* or 0).

Buying the lottery. The other possibility is that you do not start out by having a ticket but can buy one. In this case you either refuse the possibility and stay as you were (outcome = 0); or you buy the ticket and end up either with the prize *less* the purchase price, or win nothing and lose your purchase price. Your *buying price*, X_b, is the amount for which you are indifferent between buying the lottery or not. If you buy the lottery, you either win $(X^* - X_b)$ or lose this purchase price $(-X_b)$. The three possible outcomes are

$$-X_b, \quad 0, \quad (X^* - X_b)$$

The indifference statement is then properly written as

$$0 \sim (X^* - X_b, \quad P; \ -X_b)$$

Note that the buying price is not the certainty equivalent to the lottery in this case.

Comparison. The difference between buying and selling the lottery is that a person starts with a different reference point, either with or without the risk. Consequently, the person faces a different range of outcomes. Because the utility function is nonlinear, this means that the buying and selling prices are not equal.

In the buying case we have

$$U(0) = PU(X^* - X_b) + (1 - P)U(-X_b)$$

In the selling case,

$$U(X_s) = PU(X^*) + (1 - P)U(0)$$

When $U(X)$ is a nonlinear function, these two situations are quite different and thus $X_b \neq X_s$. Consequently the person measuring utility must be quite careful in how the questions are posed.

Many practical examples of buying and selling lotteries exist. A person making an investment can be considered to be buying a lottery: the investor exchanges a certain sum for a property with uncertain returns. Someone who buys insurance is, on the other hand, "selling" a lottery: in exchange for premiums of fixed amount, somebody else takes the risks.

19.7 SPECIAL DEFINITIONS

The literature on utility in operations research uses a number of phrases to describe the shape of the utility function. They are worth knowing since they are widely used. Unfortunately they are also misleading.

The first concept is that of "risk aversion" (see Section 18.3). A person is said to be "risk averse" when indifferent between a certain amount, CE, and an uncertain situation whose expected value equals $EV(L)$, where CE is less desirable than $EV(L)$. Formally

$$\text{CE} \sim (X_i, P_i; \ \ldots) \text{ for CE} < EV(L) = \sum P_i X_i$$

When people are "risk averse" their certainty equivalent to a lottery will be less beneficial than the certain amount equal to the expected value of the lottery ("less beneficial" rather than "less" because, if X is an undesirable quantity, it would be "more").

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For example, suppose there is a ticket for a 50:50 chance to win $20, as in the box in Section 18.3, with an expected value of $10. Any person who prefers to receive $9 or any CE < $10 for sure rather than have the ticket, is said to be "risk averse" under these circumstances. This means that a person's utility function would be as shown in Figures 19.1 and 19.3.

The concept of "risk aversion" is misleading because, as indicated in Section 18.3, it may have nothing to do with any attitude toward risk. The shape of the utility function may simply represent diminishing marginal utility that exists regardless of whether there is risk or uncertainty.

In practice, "risk aversion" is an alternate means of referring to concave utility functions. Yet this is not quite right either. A person may be risk averse for a choice, within the definition given above, over a range of X for which the utility function is both concave and convex, as it normally is when there are thresholds for major changes in value. See Figure 19.4.

Following in the track laid by "risk aversion," a person who chooses according to expected values is said to be *"risk neutral"*:

$$\text{CE} = EV(L) \sim (X_i, P_i, \ldots)$$

Likewise a person exhibits *"risk preference"* when preferring the lottery to some

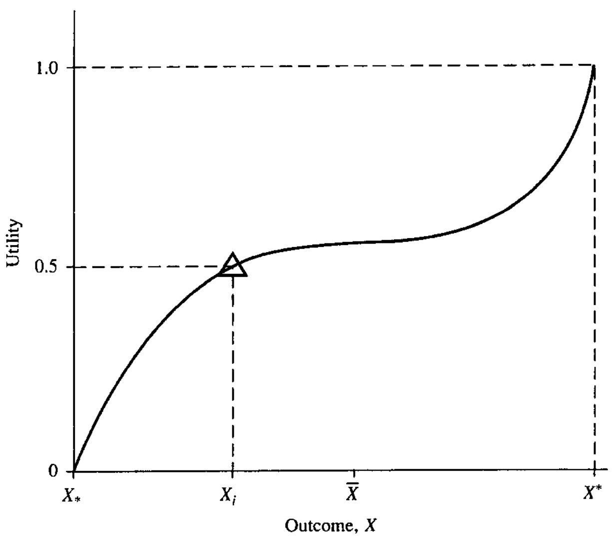

FIGURE 19.4
A person may be "risk averse" toward a lottery spanning concave and convex portions of the utility function, for example: $X_i \sim (X^*, 0.5;\ X_*)$ $X_i < EV(X)$.

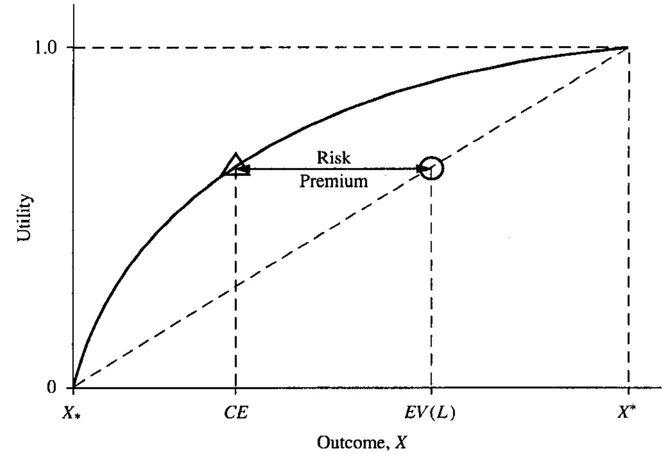

FIGURE 19.5
Definition of "risk premium," using the lottery (X^*, P; X_*).

certain amount better than the expected value. A person with "risk preference" is sometimes said to be "risk prone." This is a particularly unfortunate phrase as it suggests someone who is accident prone, which has nothing to do with the matter.

Finally, the term *"risk premium"* is used to describe the difference between the equivalent to a lottery and the expected value:

$$\text{Risk premium} = EV(L) - \text{CE}$$

Figure 19.5 illustrates the concept. As with "risk aversion" it is a misleading phrase. The distance to which it refers may simply be caused by a person's diminishing marginal utility and have nothing to do with the uncertainty.

19.8 EXAMPLE MEASUREMENT

This example replicates a process we routinely use in class to illustrate the step-by-step process for measuring utility, as presented in Section 19.5. In what follows, we illustrate the use of both the certainty equivalent and lottery equivalent/probability methods by repeating the assessment and interpretation, steps 3 and 4, for each case.

1. Defining X. We start with the assumption that we are interested in evaluating investments in different consumer items likely to interest the class, stereo sets for instance. The utility function for money is thus required. For this example, consider the range from $X_* = \$0$ to $X^* = \$1000$ (but any other can be assumed).

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2. Setting context. Building on the context of the purchase of something for enjoyment, as a stereo, it is appropriate to suggest a situation in which the person being interviewed is receiving a gift of money as a present.

My own practice is to ask the person to imagine that he or she has an eccentric relative who likes to play games—and has perhaps been influenced by a recent trip to Las Vegas. This relative supposedly offers lotteries as gifts rather than simply cash.

3. Assessment (CE). With the certainty equivalent method the person is asked to provide the certain amount which is (for them) of equal value as the lottery. We determine this by bracketing, asking the person to choose between certain amounts and the lottery until he or she is indifferent between the two.

A typical starting question is:

> "Suppose your relative offers a choice for your holiday present: either a 50:50 chance at $1000 or nothing, or a certain amount. Would you take the certain amount if it were $900?"

Most people find this easy to answer and prefer the $900.

The subsequent questions vary the certain amount in the question. First to $100, say. Again, most people find this easy; they prefer the lottery. Their indifference point must therefore lie in between $100 and $900; it is the point their preference between the lottery and the certain amount switches. To find this we try certain amounts of $700, $200, and so on until the range is really small and the person can provide the certainty equivalent directly. For this example, suppose that it is $320.

4. Interpretation (CE). The above certainty equivalent defines the equation:

$$U(320) = 0.50U(1000) + 0.50U(0)$$

With conventional definitions of the best and worst extremes for the range

$$U(1000) = 1.0 \qquad U(0) = 0$$

we obtain

$$U(320) = 0.50$$

This is easily plotted on a graph of $U(X)$ versus X.

Using the standard terminology discussed in the previous section, this utility function would be said to show "risk aversion." The "risk premium" for the ($1000, 0.5; 0) lottery would be $180, the difference between the certainty equivalent and $500, which is the monetary expectation of the stimulus lottery.

3. Assessment (LEP). Repeating the process for the lottery equivalent method, we simply alter the response suggested to the person being interviewed. The person must choose between two lotteries.

A typical starting question would thus be:

> "Suppose your relative offers a choice for your holiday present, either a 50:50 chance on getting $500 or nothing, or a 45% chance of getting $1000 or nothing?"

Notice that the probability in the second lottery, $P = 0.45$, is less than in the first. If it were larger, the second lottery should always be preferred as it would have the greater chance of the greater prize. For most people the above question is easy and they would prefer the second lottery.

Subsequent questions vary the probability in the second lottery. First to another easy choice, $P = 10\%$, and then to increasingly difficult ones such as $P = 35\%$, 15%, and so on. Finally the person will be able to supply the lottery equivalent/probability. For this example, suppose that it is 35%.

4. Interpretation (LEP). The equivalence now is

$$(1000, 0.35;\ 0) \sim (500, 0.50;\ 0)$$

which translates to the equation

$$0.35U(1000) + 0.65U(0) = 0.50U(500) + 0.50U(0)$$

The solution when the conventional values are assigned to the extremes of the range are, as indicated in Section 19.5,

$$U(500) = 2(0.35) = 0.70$$

Again, this defines a "risk averse" utility function.

5. Functional Approximation. Using the above result for the LEP assessment, we can now approximate the utility function by a formula. Taking the power function by way of example, we have, for each of the three known points,

$$U(0) = 0 = a + b(0)^c$$
$$U(500) = 0.70 = a + b(500)^c$$
$$U(1000) = 1.0 = a + b(1000)^c$$

The first equation implies $a = 0$. This means that we can get an expression in c alone by dividing the last two equations. The result is

$$0.70 = (0.5)^c$$

so that

$$c = 0.514$$

and then

$$b = (1000)^{-c} = 0.0287$$

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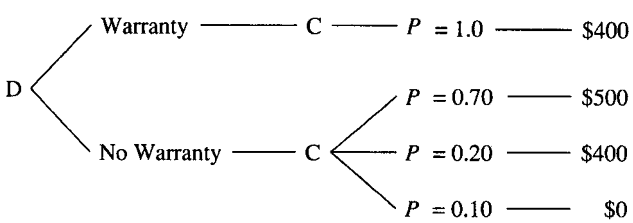
The power approximation to this utility function is

$$U(X) = 0.0287X^{0.514} \approx 0.03X^{0.5}$$

19.9 USE IN DECISION ANALYSIS

This section illustrates how utility functions are integrated in decision analysis. The basic idea is simple: substitute the utility of the outcome for the outcome and do the analysis. The result will be the preferred decision once a person's utility is considered.

Consider that the person whose utility we assessed and fitted in Section 19.8 has in fact received a $1000 holiday present. Imagine further that this person intends to buy a $500 stereo and has to decide whether or not to buy the $100 service contract that would guarantee all repairs at no further cost. With this warranty, the total cost of the system would be $600 so our friend would end up with $400 left. Without it, the total cost depends on the reliability of the system. The records show that there is a 70% chance of no problems and no extra cost, a 20% chance of a $100 repair, and a 10% chance that the whole system must be replaced at an extra cost of $500. In that case, our friend would have no money left. The decision tree for our problem, stated in terms of money left, is

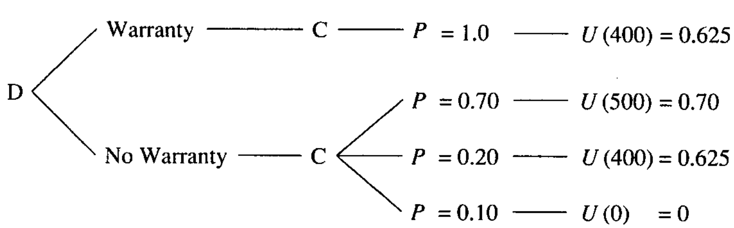


Looking only at the money outcomes, it would seem better not to have the warranty. The expected money left would be

$$\text{Expected \$, No Warranty} = 430$$

This is more than what our friend would have with the warranty.

Integrating utility in the analysis, we transform the decision tree by substituting the utilities. Thus



The better choice, considering utility, is to take the warranty. Indeed,

$$\text{Utility, No Warranty} = 0.7(0.7) + 0.2(.625) = 0.615$$

This is less than the utility when one has the warranty. As should be expected, the person who is "risk averse" similarly may be expected to weigh safer choices more favorably in practice.

The purpose of measuring a person's utility in a controlled setting is to provide insight into how that individual would choose. The utility function enables the analyst to incorporate that person's values into complex situations, and to determine which strategies are best for that person—a task that person would normally not be able to do, unless skilled in decision analysis.

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PROBLEMS

19.1. *Utility Manipulation I*

(*a*) Given the lottery ($16, 0.75; $0) and the utility function $U(X) = (900 + 100X)^{1/2}$, determine the lottery's expected value; expected utility; selling price; buying price. Does $U(X)$ show "risk aversion"?

(*b*) Repeat (*a*) for the lottery (100, 0.5; 0) and $U(X) = (100 + 3X)^{1/2}$.

19.2. *Utility Manipulation II*

(*a*) A person is found to have the utility function: $U(X) = (400 + 100X)^{1/2}$, X in $. For each of the lotteries ($21, 0.5; −$3) and ($16, 0.5; $0) find the expected value, expected utility, and selling price.

(*b*) Repeat (*a*) using the utility function $U(X) = (400 + 20X)^{1/2}$ and the lottery ($125, 0.4; $0).

19.3. *Utility Manipulation III*

Ms. Thomas has the utility function $U(X) = [(100 + X)/200]^{1/2}$. For what probability P would she have a selling price of −50 for the lottery (28, P ; −68)?

19.4. *Probability Equivalents*

Given the following utility function:

X	:	−100	−90	−80	−65	−50	−35
$U(X)$	:	0	3	4.5	6	7	8
X	:	−20	0	20	45	70	110	150
$U(X)$	:	9	10	11	12	13	14	15

(*a*) Calculate the probabilities P_1 and P_2 such that 45 is indifferent to the lotteries (20, P_1; 150) and (−20, P_2; 110).

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(*b*) Calculate the selling price and risk premiums of the lotteries: (−90, 0.5; −20); (−65, 0.8; −20); (−100, 0.2; −50, 0.3; −20, 0.4; 70).
(*c*) Calculate the buying price for: (70, 0.5; 150); (150, 0.4; 45).

19.5. *Jones*
Suppose Jones tells you his selling prices for the lotteries (−$40, 0.2; $70) and (−$40, 0.75; $30) are 30 and 0 respectively. Note, Jones is not "risk averse" between −$40 and $30. He is "risk averse" between $30 and $70.
(*a*) Plot a plausible utility function for Jones indicating key points.
(*b*) If $U(70) = 25$ and $U(30) = 15$, what would $U(0)$ equal?

19.6. *Table*
Consider the table below. Does this utility function show the respondent to be "risk averse," "risk seeking," or "risk neutral"?

Travel time (min)	Utility
0	1.0
10	0.8
20	0.5
30	0

Explain.

19.7. *Utility Functions*
Which of the following utility functions represent an "expected value maximizer," a "risk positive" individual, or a "risk averse" individual?
(*a*) $U(X) = X$
(*b*) $U(X) = 2X + 4$
(*c*) $U(X) = -e^{-3x} - e^{-x}$
(*d*) $U(X) = X^2$
(*e*) $U(X) = X^{0.5}$

19.8. *Workstations*
Lee O'Tard, President of Munskin clothes, needs to acquire computer workstations to design new fashions. The salient criteria are cost and speed. Through interviews, Lee indicates that:

- where costs are concerned: $4000 ~ ($6000, 0.5; $1000)
- for megahertz of speed: 10 ~ (24, 0.2; 4) and 18 ~ (24, 0.75; 10)

(*a*) Plot the utility function for cost.
(*b*) Is Lee "risk averse" for money?
(*c*) Plot the utility function for speed.

19.9. *Fire Chief*
A fire chief who wants to minimize response time to answering fire calls indicates indifference between 1.6, 2.25, and 0.5 min for sure and the following lotteries, respectively:

(2.5 min, 0.4; 0 min) (2.5 min, 0.5; 1.6 min) (1.6 min, 0.75; 0 min)

Is the decisionmaker "risk averse" or not? Explain.

19.10. *Charles Chancy*
Suppose you wanted to construct a utility function for Charles who tells you that 320 and 850 are respectively his certain monetary equivalents for the lotteries ($600, 0.5; $0) and ($1000, 0.5; $600). Assuming Charles exhibits "risk preference," draw a graph that indicates how the above information restricts Charles' utility function in the region $X = 0$ to $X = 1000$.

19.11. *Lottery Construction*
Write the lotteries defined by the following statements:
(*a*) A manager is willing to invest no more than $4 million in a project that is equally likely to gross $10 million or nothing.
(*b*) Another manager is willing to sell for no less than $1 million the plans to a new process that would make $3 million with probability of 0.7 and nothing with a probability of 0.3.
(*c*) A third manager is willing to pay no more than $2 million to avoid a 75% chance of losing $5 million and a 25% chance of making $1 million.

19.12. *Smith*
You are told that Smith could be willing to sell for no less than $20 the lottery ($50, $\frac{2}{3}$; −$100); to pay up to $30 to get the chance to play the lottery ($50, $\frac{3}{4}$; −$70); and has a "risk premium" of −$22.5 for the lottery ($50, 0.05; $0, 0.10; −$100, 0.85).
Sketch Smith's utility function over the range for which you have information. Clearly label axes and the coordinates of the important points.

19.13. *Bob*
Given that Bob is willing to sell the lottery (100, 0.8; −50) for no less than $50; to pay up to $40 to avoid the lottery (50, 0.25; −50); and to buy the lottery (75, 0.5; −15) for no more than $25,
(*a*) Sketch Bob's utility function between −$50 and $100.
(*b*) What is Bob's selling price and risk premium for (100, 0.75; −40)?

19.14. *Cathy*
Suppose that Cathy is willing to pay a maximum of $50 to get out of playing the lottery ($100, 0.5; −$100), to pay up to $10 to buy the lottery ($110, 0.5; −$40), and to sell for no less than $60 the lottery ($100, $\frac{2}{3}$; 0). Assuming that Cathy is consistently "risk averse":
(*a*) Sketch her utility function between −$100 and $100.
(*b*) For Cathy to be consistent, what should her selling price be for the lottery: ($0, $\frac{2}{3}$; −$100)?

19.15. *John*
John would be willing to sell the lottery ($500, 0.5; −$100) for no less than $100. He would pay up $50 to play the lottery ($150, 0.6; −$50). He would also give up the lottery ($500, 0.6; $100) for no less than $300.
(*a*) Write John's indifference statements for the above three lotteries.
(*b*) Sketch his utility function. Clearly label the axes and points.
(*c*) What factors may account for a utility function that is "risk averse"? Classify John's utility function as either "risk averse" or "risk seeking."

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19.16. *Petra*

Petra would sell the lottery ($80, $\frac{2}{3}$; −$20) for $70; the lottery ($70, 0.6; $0) for $50; and would pay $20 for ($100, 0.1; $0).

(*a*) Write her indifference statements for the three lotteries.
(*b*) Sketch her utility function, clearly labelling axes and points.
(*c*) What factors may account for a utility function that is "risk averse"? Classify her utility function as "risk averse" or "risk seeking."

19.17. *Andy*

Given the lottery: (100, 0.5; 0) and the utility function $U(X) = (100 + 3X)^{1/2}$, calculate its expected value; expected utility; selling price; and buying price.

19.18. *Georgia*

By interviewing Georgia you find that she would sell the following lottery for $50 : ($100, 0.9; −$80); would pay $50 to get ($100, 0.8; −$30) and $60 to avoid ($0, $\frac{1}{3}$; −$80).

(*a*) Write Georgia's indifference statements for the three lotteries.
(*b*) Sketch the utility function, clearly labelling axes and points.
(*c*) For Georgia to be consistent, find the probability for the statement $0 ~ ($50, *P*; −$60).

19.19. *Meredith*

Meredith's utility function for money is $U(X) = (5 + .01X)^{1/2}$ for all X from −500 to 1000 dollars. Calculate Meredith's selling price, buying price, and risk premium for the lottery ($900, 0.4; $400).

19.20. *David*

David is willing to sell the lottery ($40, 0.6; 0) for a minimum of $22. He will pay a maximum of $10 for the lottery ($32, $\frac{5}{8}$; −$20). David is always "risk averse" and prefers increasing amounts of money.

(*a*) Draw a graph that shows how the above information restricts David's utility function in the region from −$30 to $40.
(*b*) David's utility function can be approximated by a positive linear transformation of $U(X) = -e^{-0.01x}$. Write an expression for his utility function with $U(40) = 1$ and $(U - 30) = 0$. (Hint: $e^{0.4} = 1.5$.)

19.21. *Toothpick*

Your friend who has made a fortune with the invention of the piezoelectric toothpick asks your advice in planning some investments. By interview you find that your friend:

- Would spend up to $3M to develop a new automotive emission control, which would, with probability 0.50, be worth $10M.
- Has a new invention, a bike lock, that will make $7M except if (one chance in five) it does not catch on; and would be willing to sell the rights to the lock for $5M or more.
- Has a batch of piezoelectric toothpicks with potential malfunctions. If they actually fail, penalties would be $3M. There is a 60% chance of this and your friend would spend up to $2M to correct the defects.

(*a*) Formulate the above as indifference statements.
(*b*) Sketch your friend's utility function.
(*c*) List 2 questions you could ask to test your friend's consistency, and indicate what answers would be appropriate.

19.22. *Vegas Bus*

The mayor of Vegas has to decide whether to spend $1M on a bus experiment along the entertainment strip. The local gamblers estimate that there is a 50-50 chance that the system's total revenues will be $0.2M, a 0.3 chance of breaking even, and a 0.2 chance of $1.4M profits. Your evidence of the mayor's attitude is her behavior last year on a $0.8M budget. She was then prepared:

- To invest the city's money in a "cavalcade of stars" which was 4:1 (0.2 chance of success) to profit $1.2M; otherwise Vegas would lose the whole budget.
- To trade, when approached by the governor's office intent on dispersing patronage, a grant of $0.8M for the opportunity to host a national convention equally probable to have $1.2M or zero profits.

(*a*) Draw the mayor's utility function, stating any necessary assumptions.
(*b*) You now find that last month the mayor decided to spend $0.9M on a beauty contest that would either gross $1.7M (probability 0.4) or $0.1M (probability 0.6). Is the mayor consistent in her general attitudes toward risk?
(*c*) What should she do about the bus experiment?

19.23. *International Exploitation*

The record of International Exploitation, Inc. indicates that the following decisions represent its attitude toward risk-taking:

- If a project is equally likely to boom (revenue = $8M) or bust (revenue = $0), I. E. will implement the project if the project costs $2M or less.
- If a smaller project is equally likely to succeed (revenue = $3.5M) or bust (revenue = $0), I. E. will not spend more than $1.5M on the project.
- The company seems indifferent among any of the three choices: $0.5M certain or ($2M, 0.4; $0) or ($6M, 0.3; $0,0.2; −$0.5M, 0.5).
- Corporate policy indicates a willingness to spend as much as $0.5M to avoid a 20% chance of losing $2M.

(*a*) Construct the lotteries outlined.
(*b*) Approximate and plot I. E.'s utility function (seven points can be inferred from the data).
(*c*) Assuming I. E. is consistent, for what cash payment would I. E. sell the lottery ($6M, $\frac{1}{3}$; −$1.5M)?
(*d*) For what probability would I. E. be indifferent between −$0.5M and ($2M, P; −$1.5M)?

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