Production Functions

Outline

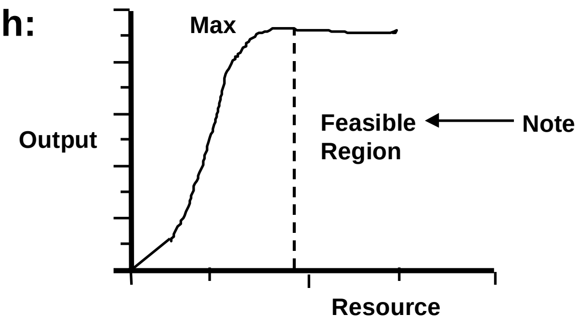
1. Definition
2. Technical Efficiency
3. Mathematical Representation
4. Characteristics

Production Function - Basic Model for Engineering

- **Definition:**
  - Represents technically efficient transformation of physical resources into products
- **Example:**
  - Use of aircraft, pilots, fuel to carry cargo, passengers
- **Focus on one-dimensional output**

Technical Efficiency

- **Definition:**
  - **Maximum product from a given set of resources $X = X_1 \dots X_n$**
- **Graph:**



Mathematical Representation - General

- **Two Possibilities**
- **Deductive**
  - **Choose convenient equation**
  - **Fit data to equation (as best as you can)**
  - **Advantage - ease of use**
  - **Disadvantage - poor accuracy**
- **Inductive**
  - **Synthesize system model from knowledge of details**
  - **Advantage - accuracy**
  - **Disadvantage - difficult to use**

contrast

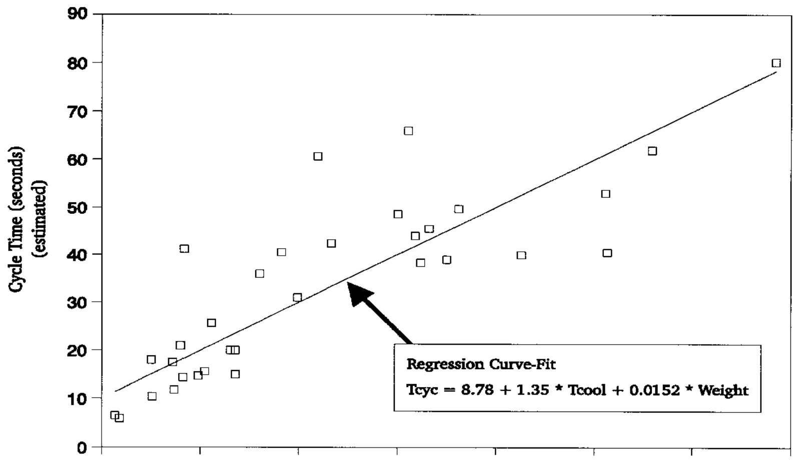
Mathematical Representation - Deductive

- Cobb-Douglas
  - $Y = a_0 \pi X_i^{a_i} = a_0 X_i^{a_i} \dots X_n^{a_n}$
  - Easy interpretation: 'a_i' are physically significant
  - Easy statistical estimation (linear least squares) $\log Y = a_0 + \Sigma a_i \log X_i$
- Translog
  - $\log Y = a_0 + \Sigma a_i \log X_i + \Sigma\Sigma a_{ij} \log X_i \log X_j$
  - More subtle, more realistic

Mathematical Representation - Inductive

- "Engineering models" of PF
- Analytic expressions
    - for force fields
    - ex: transport in fluid, river
  - Disadvantage: rare
- Detailed simulation
  - Disadvantages
    - time consuming
    - need for data
    - expensive

Cooling Time, Part Weight, and Cycle Time Correlation

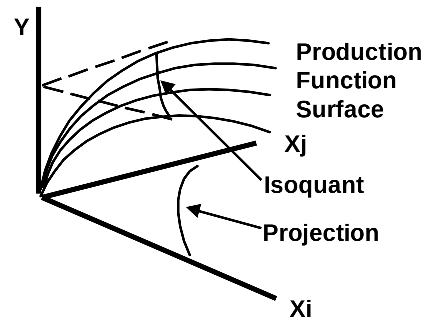


PF: Characteristics

- Isoquants
- Marginal Products
- Marginal Rates of Substitution
- Returns to Scale
- Convexity of Feasible Region

Characteristic: Isoquants

- **Definition:**
  - **Locus (contour) of equal product on production function**
- **Graph:**



Characteristic: Isoquants (cont'd)

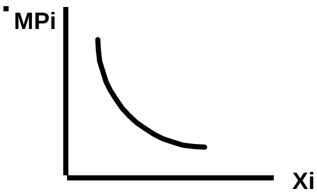
- **Note:**
  - **All (the many) points on isoquant are technically efficient, therefore, no technical basis for choice from among them**
- **Economics (values) are decisive**

Characteristic: Marginal Products

- **Definition:**
  - **Marginal Product is the change in output as only one resource changes**

$$MP_i = \partial Y / \partial X_i$$

- **Graph:**



"Law of Diminishing Marginal Products"

Characteristic: Marginal Products (cont'd)

- **Math:**

$$Y = a_0 X_1^{a_1} \ldots X_i^{a_i} \ldots X_n^{a_n}$$

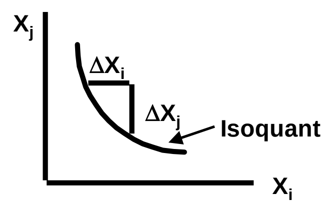
$$\partial Y / \partial Xi = (ai/Xi)Y$$

$$= f\ (Xi^{a_i - 1})$$

Diminishing Marginal Product if $a_i < 1.0$

Characteristic: Marginal Rate of Substitution

- **Definition:**
  - **Rate at which one resource must substitute for another so that product is constant**
- **Graph:**



Characteristic: Marginal Rate of Substitution (cont'd)

- **Math:**

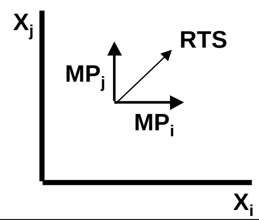
  since $\Delta X_i MP_i + \Delta X_j MP_j = 0$
  (no change in product)

  then $MRS_{ij} = \Delta X_i / \Delta X$
  $= - MP_j / MP_i = - (a_j / a_i)(X_i / X_j)$

  MRS is "slope" of isoquant

Characteristic: Returns to Scale

- **Definition:**
  - Ratio of rate of change in Y to rate of change in all <u>X</u> (each X_i changes by same factor)
- **Graph:**
  - Directions in which the rate of change in output is measured for MP and RTS



Characteristic: Returns to Scale (cont'd)

- **Math:**

$$Y' = a_0 \pi X_i^{a_i}$$
$$Y'' = a_0 \pi (sX_i)^{a_i}$$
$$= Y'(s)^{\Sigma a_i}$$

$$RTS = (Y''/Y')/s = s^{(\Sigma a_i - 1)}$$

Y''/Y' = % increase in Y
if $Y''/Y' > s$ => IRTS

Increasing returns to scale if $\Sigma a_i > 1.0$

Increasing Returns to Scale

- **Practical Importance:**
  - **IRTS means that bigger units are much more productive than small ones**

- **IRTS => concentration of production toward ever larger units**

Increasing Returns to Scale (cont'd)

- **Practical Occurrence:**
  - **Frequent!**
  - **Generally where**
    - **Product = *f* (volume) and**
    - **Resources = *f* (surface)**
  - **Example:**
    - **ships**
    - **pipelines, cables**
    - **chemical plants**
    - **etc.**

Characteristic: Convexity of Feasible Region

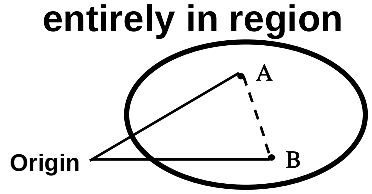
- **Definition:**
  - **Region is convex if it has no reentrant corners**

- **Graph:**

Characteristic: Convexity of Feasible Region (cont'd)

- **Math: If A, B are two vectors to any 2 points in region**

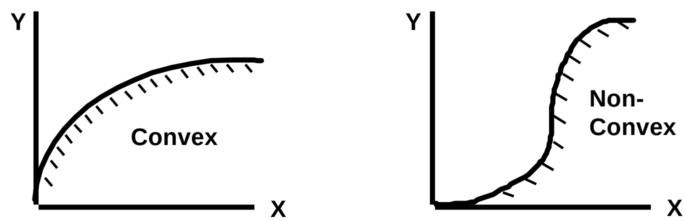
  Convex if all
  $T = KA + (1-K)B \quad 0 \leq K \leq 1$
  entirely in region



- **Usefulness: Optimization is much easier if feasible region is convex**

PF: Convexity of Feasible Region

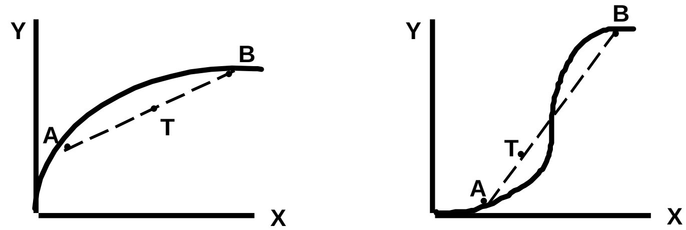
- F.R. is convex if no reentrant corners



- Convexity => Easy Optimization

PF: Convexity of Feasible Region (cont'd)

- Test for Convexity: Given A,B on PF
  If T = KA + (1-K)B $0 \leq K \leq 1$
  Convex if all T in region



- Cobb-Douglas: $a_i \leq 1.0$ and $\Sigma a_i \leq 1.0$