Constrained Optimization

Outline

Unconstrained Optimization (Review)

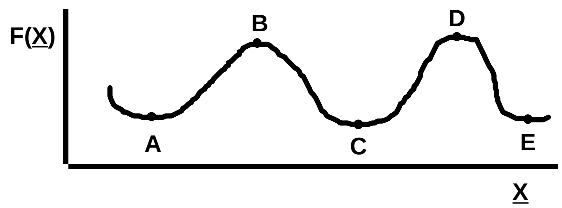
Constrained Optimization

1. Approach
2. Equality constraints
   - Lagrangeans
   - Shadow prices
3. Inequality constraints
   - Kuhn-Tucker conditions
   - Complementary slackness

Unconstrained Optimization (1)

- **Definitions:**
  - Optimization = Maximum of desired quantity
    = Minimum of undesired quantity
  - Objective Function = Expression to be optimized
    = Z(X)
  - Decision Variables = Variables about which we can make decisions
    = X

Unconstrained Optimization (2)



- **By calculus:**

  If F(X) continuous, analytic:
  Condition for maxima and minima

  $\partial F(X) / \partial X_i = 0 \qquad \forall_i$

Unconstrained Optimization (2) (cont'd)

- **Secondary conditions:**

  $\partial^2 F(\underline{X}) / \partial X_i^2 < 0$ = > Max (B,D)

  $\partial^2 F(\underline{X}) / \partial X_i^2 > 0$ = > Min (A,C,E)

Unconstrained Optimization (3)

- **Example: Housing insulation**

  $F(x) = K_1 / x + K_2x$

  Cost = Fuel cost + Insulation cost

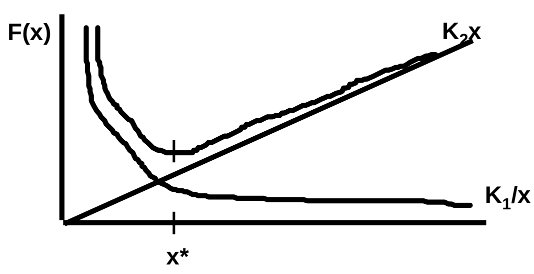
  x = Thickness of insulation

  $\partial F(x) / \partial x = 0 = -K_1 / x^2 + K_2$

  $\Rightarrow x^* = \{K_1 / K_2\}^{1/2}$

  (starred quantities are optimal)

Unconstrained Optimization (3) (cont'd)



Constrained Optimization

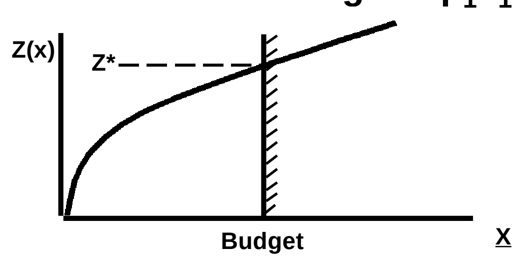
- Approach

  To solve situations of
  increasing complexity
  (equality, inequality constraints) ...

  Transform more difficult situation
  into one we know how to deal with

Equality Constraints

- Example: Best use of budget
- Maximize: Output = $Z(\underline{X}) = a_o x_1^{a1} x_2^{a2}$
- Subject to (s.t.):
  Total costs = Budget = $p_1 x_1 + p_2 x_2$



Note: $\partial Z(\underline{X}) / \partial \underline{X} \neq 0$ at optimum

Lagrangean Method (1)

Transforms equality constraints into unconstrained problem

Start with:

Opt: F(x)

s.t.: $g_j(\underline{x}) = b_j \Rightarrow g_j(\underline{x}) - b_j = 0$

Get to:

$L = F(\underline{x}) - \Sigma_j \lambda_j[g_j(\underline{x}) - b_j]$

λj = Lagrangean multipliers
(these are unknown quantities which must be solved for)

Note: $[g_j(\underline{x}) - b_j] = 0$ so that opt $F(\underline{x})$ = opt L

Lagrangean Method (2)

- **To optimize L:**

$$\partial L / \partial x_i = 0 \qquad \forall_i$$
$$\partial L / \partial \lambda_j = 0 \qquad \forall_i$$

- **Example:**

Opt: $F(\underline{x}) = 6x_1x_2$

s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18$

$$L = 6x_1x_2 - \lambda(3x_1 + 4x_2 - 18)$$

$$\partial L / \partial x_1 = 6x_2 - 3\lambda = 0$$
$$\partial L / \partial x_2 = 6x_1 - 4\lambda = 0$$
$$\partial L / \partial \lambda_i = 3x_1 + 4x_2 - 18 = 0$$

Lagrangean Method (3)

- so that: $\lambda = 2x_2 = 1.5x_1$
  $x_2 = 0.75x_1$
  $3x_1 + 3x_1 - 18 = 0 \Rightarrow x_1^* = 3$
  $x_2^* = 2.25$ $\lambda^* = 4.5$
  $F(x)^* = 40.5$

Shadow Prices

- Definition:
  - Shadow Price = Rate of change of objective function per unit change of constraint
    $= \partial F(\underline{x}) / \partial b_j$
  - This is meaning of Lagrangean multiplier
    $SP_j = \partial F(\underline{x})^* / \partial b_j = \lambda_j$
  - Naturally, this is an instantaneous rate

Shadow Prices (cont'd)

- **Example:**

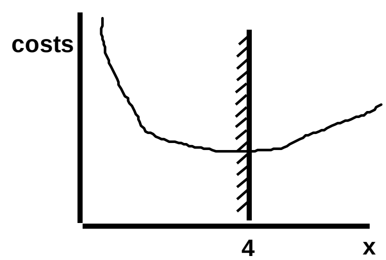
  Opt: $F(\underline{x}) = 6x_1x_2$
  s.t.: $g(\underline{x}) = 3x_1 + 4x_2 = 18.1$

  $\Rightarrow x_1^* = 18.1/6 \qquad x_2^* = 18.1/8$
  $F(x)^* = 6(18.1/6)(18.1/8) = 40.95$
  $\Delta F(x) = 40.95 - 40.5 = 0.45 = \lambda^* (0.1)$

Inequality Constraints

- **Example: Housing insulation**

  Min: Costs = $K_1 / x + K_2x$
  s.t.: $x \geq 4$ (minimum thickness)



Inequality Constraints (1)

- Kuhn-Tucker conditions
- Transform inequalities into equalities, then proceed as before
- Again, introduce new variable

Inequality Constraints (2)

- This is "slack variable", s_j

  $g_j(\underline{x}) \le b_j \quad => \quad g_j(\underline{x}) + s_j^2 = b_j$

  $g_j(\underline{x}) \ge b_j \quad => \quad g_j(\underline{x}) + s_j^2 = b_j$

  start from:

  opt: $F(\underline{x})$

  s.t.: $g_j(\underline{x}) \le b_j$

  get to:

  $L = F(\underline{x}) - \Sigma_j \lambda_j [g_j(\underline{x}) + s_j^2 - b_j]$

Inequality Constraints (3)

- **Complementary slackness**
- **The optimality conditions are:**

  $\partial L / \partial x_i = 0$

  $\partial L / \partial \lambda_j = 0$

  plus: $\partial L / \partial s_j = 2 s_j \lambda_j = 0$
- **These new equations imply:**

  $s_j = 0$ $\lambda_j \neq 0$

  or

  $s_j \neq 0$ $\lambda_j = 0$

  This is the "complementary slackness"

Inequality Constraints (4)

- **Interpretation:**
- **If there is slack on bj,**
  (i.e. more than enough of it)

  => No value to objective function to having more: $\lambda_j = \partial F(\underline{x}) / \partial b_j = 0$

  If $\lambda_j \neq 0$, then all available b_j used

  => $s_j = 0$