SMA ADVANCED MATERIALS PROGRAM
MATERIALS SELECTION, DESIGN AND ECONOMICS
SMA 5103 & MIT 3.57
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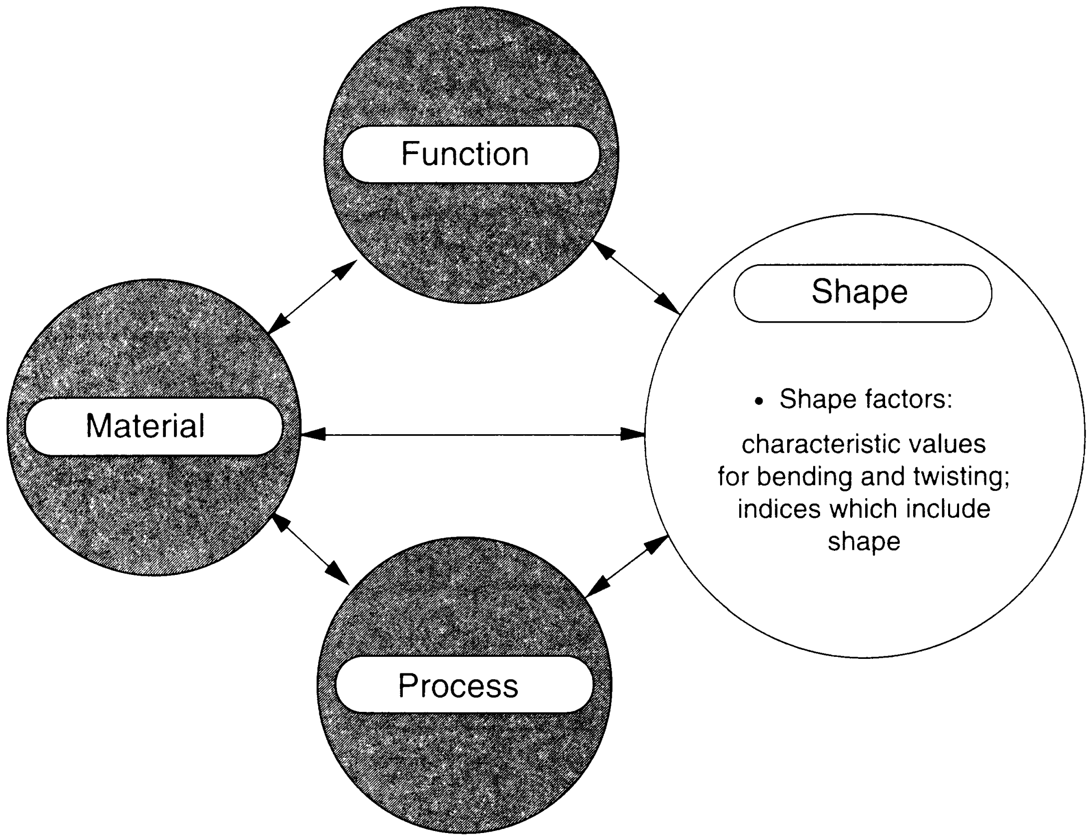
MATERIALS SELECTION IN MECHANICAL DESIGN

PART 4

L. Anand

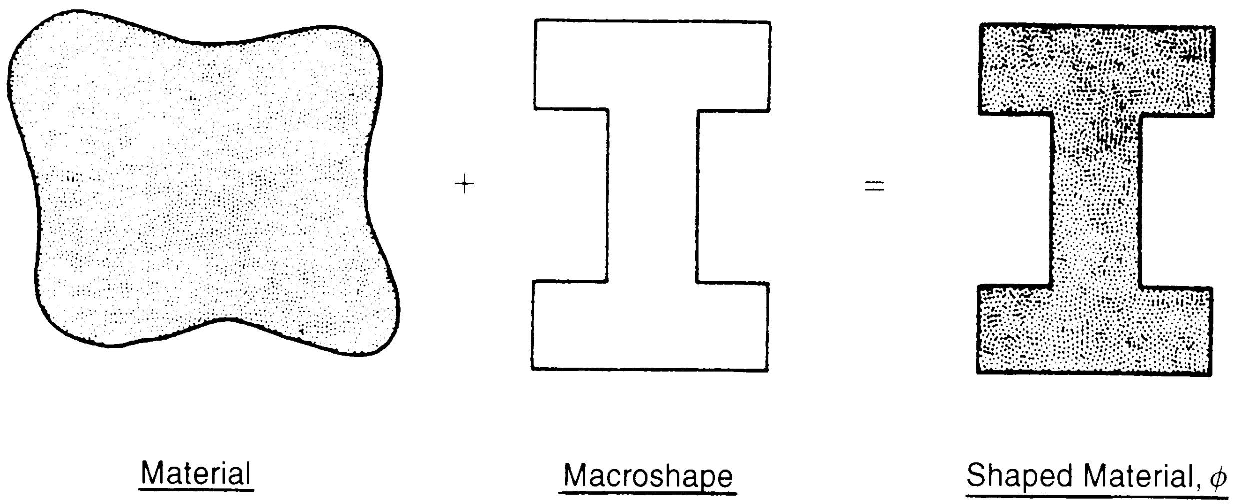
SELECTION OF MATERIAL AND SHAPE



Section shape is important for certain modes of loading. When shape is a variable a new term, the shape factor, appears in some of the material indices: they then allow optimum selection of material and shape.

- A **material** has properties but no shape.

- A **component** is a material made into a shape:



Mechanical efficiency is obtained by combining material with macroscopic shape. The shape is characterized by a dimensionless shape factor, φ.

- Shaped sections carry bending, torsional and axial compressive loads more efficiently than solid sections do.

- Shaped section $\equiv$ cross-section is in the form of a tube, an I-section, a boxed-section, etc.

- Efficient $\equiv$ for given external loading, the shaped section uses less material, and is therefore lighter.

If materials are available in different section shapes, and if for a given design the section shape matters, then the material selection problem becomes

- How to choose from amongst the vast range of materials, and the section shapes in which they are available (or potentially could be made into), the one one which maximizes performance.

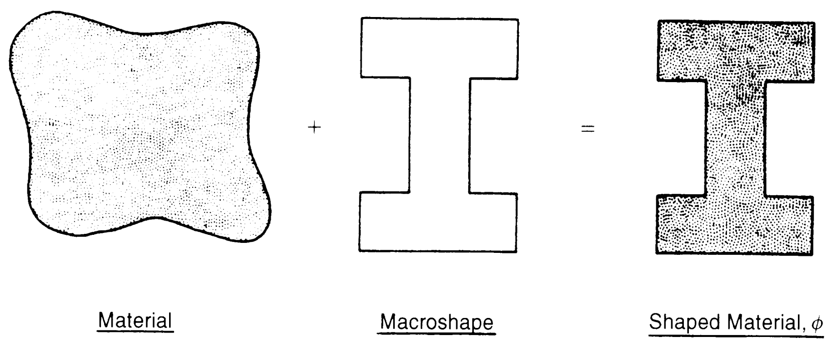
To answer this question, we introduce

- **Shape factors**: these are dimensionless numbers which characterize the effficiency of shaped sections.

This allows the definition of "material indices" which include not only material properties but also the shape factors.

- A **material** has properties but no shape.

- A **component** is a material made into a shape:



Mechanical efficiency is obtained by combining material with macroscopic shape. The shape is characterized by a dimensionless shape factor, φ.

- **Shape factors**: these are dimensionless numbers which characterize the effficiency of shaped sections.

We shall introduce the following shape factors:

ϕ_B^e	Shape factor for elastic bending of beams
ϕ_T^e	Shape factor for elastic twisting of shafts
ϕ_B^f	Shape factor for failure/yielding of beams in bending
ϕ_T^f	Shape factor for failure/yielding of shafts in torsion

Elastic Bending of a Beam, ϕ_B^e

The **bending stiffness** of a beam of length l, made from a material with Young's modulus E is

$$S_B = \frac{F}{\delta} = \frac{C_1 EI}{l^3}.$$

The shape enters through the **second moment of area**

$$I = \int_A y^2 \mathrm{d}A.$$

Let

$$S_B^o = \frac{C_1 E I^o}{l^3}$$

denote the stiffness of the beam of the same length l, made from the same material with Young's modulus E, but with a **solid circular cross-section** with an second moment of area

$$I^o = \frac{\pi r^4}{4} = \frac{A^2}{4\pi}$$

such that the cross-sectional area A of the beam with the solid circular section is the same as the area of the beam with a more general cross-section.

Then the shape factor for elastic bending of beams is defined as

$$\phi_B^e \equiv \frac{S_B}{S_B^o} \quad \Rightarrow \quad \phi_B^e = \frac{I}{I^o} = \frac{4\pi I}{A^2}$$

or

$$\boxed{\phi_B^e = \frac{4\pi I}{A^2} \quad \text{dimensionless}}$$

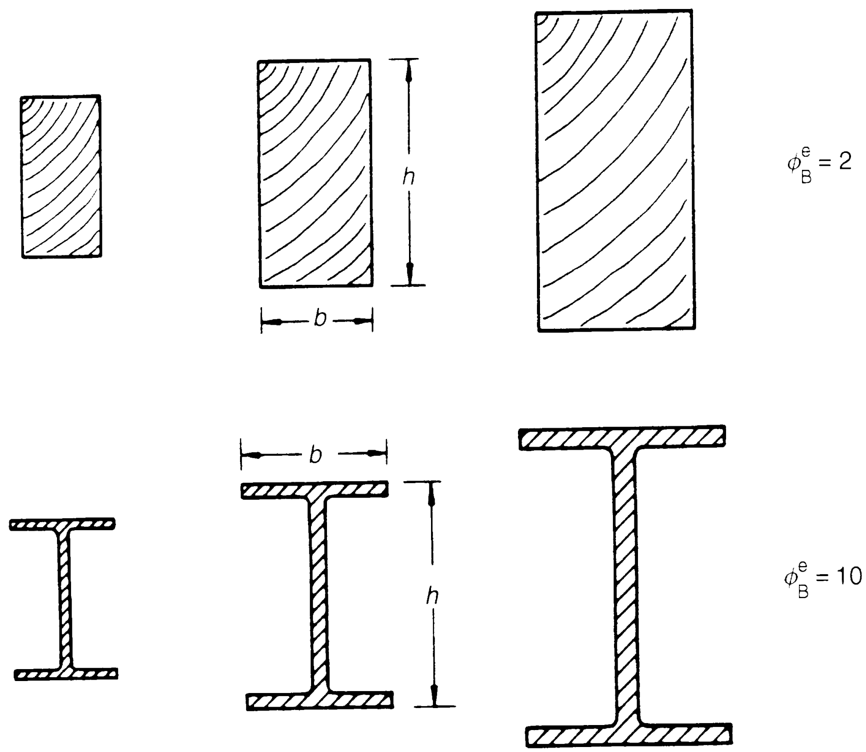
As an example consider a beam with a rectangular cross-section of width b and height h. For this cross-section $I = \dfrac{bh^3}{12}$ and $A = bh$. Hence

$$\phi_B^e = \frac{4\pi I}{A^2} = \frac{4\pi bh^3}{12b^2h^2},$$

or for a beam with a rectangular cross-section

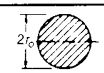
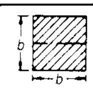
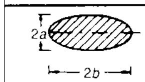
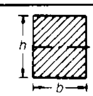
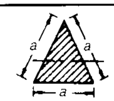
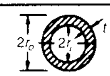
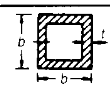
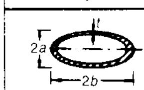
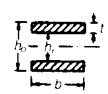
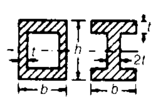
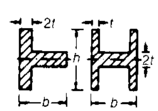
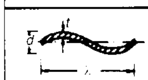
$$\boxed{\phi_B^e = \frac{\pi h}{3b}}$$

Note that since ϕ_B^e is dimensionless, for the same section shape, beams with different sized cross-sectional areas have the same value of ϕ_B^e:

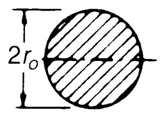
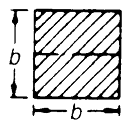
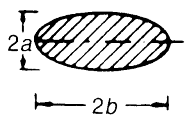
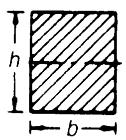
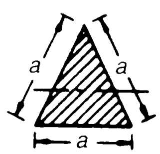
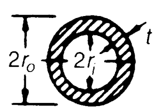


A set of rectangular sections with $\phi_B^e = 2$, and a set of I-sections with $\phi_B^e = 10$. Members of a set differ in size but not in shape.

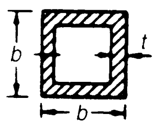
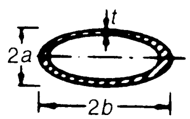
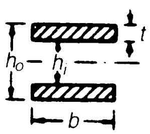
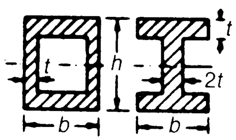
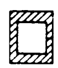
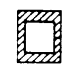
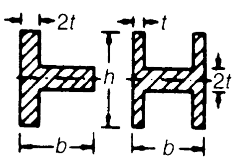
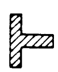
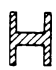
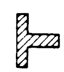
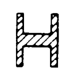
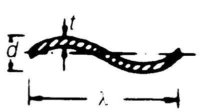
Moments of areas of sections for common shapes

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4} r^4$	$\frac{\pi}{2} r^4$	$\frac{\pi}{4} r^3$	$\frac{\pi}{2} r^3$
	b^2	$\frac{b^4}{12}$	$0.14 b^4$	$\frac{b^3}{6}$	$0.21 b^3$
	πab	$\frac{\pi}{4} a^3 b$	$\frac{\pi a^3 b^3}{(a^2 + b^2)}$	$\frac{\pi}{4} a^2 b$	$\frac{\pi a^2 b}{2}$ $(a < b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3 h}{3}\left(1 - 0.58\frac{b}{h}\right)$ $(h > b)$	$\frac{bh^2}{6}$	$\frac{b^2 h^2}{3h + 1.8b}$ $(h > b)$
	$\frac{\sqrt{3}}{4} a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2 - r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4 - r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2}(r_o^4 - r_i^4)$ $\approx 2\pi r^3 t$	$\frac{\pi}{4r_o}(r_o^4 - r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{2r_o}(r_o^4 - r_i^4)$ $\approx 2\pi r^2 t$
	$4bt$	$\frac{2}{3} b^3 t$	$b^3 t\left(1 - \frac{t}{b}\right)^4$	$\frac{4}{3} b^2 t$	$2b^2 t\left(1 - \frac{t}{b}\right)^2$
	$\pi(a + b)t$	$\frac{\pi}{4} a^3 t\left(1 + \frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2} t}{(a^2 + b^2)}$	$\frac{\pi a^2 t}{4}\left(1 + \frac{3b}{a}\right)$	$2\pi t(a^3 b)^{1/2}$ $(b > a)$
	$b(h_o - h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3 - h_i^3)$ $\approx \frac{1}{2} bth_o^2$	—	$\frac{b}{6h_o}(h_o^3 - h_i^3)$ $\approx bth_o$	—
	$2t(h + b)$	$\frac{1}{6} h^3 t\left(1 + \frac{3b}{h}\right)$	□: $\approx \frac{2tb^2h^2}{h + b}$; I: $\frac{2}{3} bt^3\left(1 + \frac{4h}{b}\right)$	$\frac{h^2 t}{3}\left(1 + \frac{3b}{h}\right)$	□: $2tbh$; I: $\frac{2}{3} bt^2\left(1 + \frac{4h}{b}\right)$
	$2t(h + b)$	$\frac{t}{6}(h^3 + 4bt^2)$	H: $\frac{t^3}{3}(8b + h)$; ⊢: $\frac{2}{3} ht^3\left(1 + \frac{4b}{h}\right)$	$\frac{t}{3h}(h^3 + 4bt^2)$	H: $\frac{t^2}{3}(8b + h)$; ⊢: $\frac{2}{3} ht^2\left(1 + \frac{4b}{h}\right)$
	$t\lambda\left(1 + \frac{\pi^2 d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

Values for the four shape factors

Section shape	Stiffness ϕ_B^e	Stiffness ϕ_T^e	Strength ϕ_B^f	Strength ϕ_T^f
	1	1	1	1
	$\frac{\pi}{3} = 1.05$	0.88	$\frac{2}{3}\sqrt{\pi} = 1.18$	0.74
	$\frac{a}{b}$	$\frac{2ab}{(a^2 + b^2)}$	$\sqrt{\frac{a}{b}}$	$\sqrt{\frac{a}{b}}$ $(a < b)$
	$\frac{\pi}{3}\frac{h}{b}$	$\frac{2\pi}{3}\frac{b}{h}\left(1 - 0.58\frac{h}{b}\right)$ $(h > b)$	$\frac{2}{3}\sqrt{\pi}\left(\frac{h}{b}\right)^{1/2}$	$\frac{2}{3}\sqrt{\pi}\frac{(b/h)^{1/2}}{(1 + 0.6b/h)}$ $(h > b)$
	$\frac{2\pi}{3\sqrt{3}} = 1.21$	$\frac{2\pi}{5\sqrt{3}} = 0.73$	0.77	0.62
	$\frac{r}{t}$	$\frac{r}{t}$	$\left(\frac{2r}{t}\right)^{1/2}$	$\left(\frac{2r}{t}\right)^{1/2}$

(continued overleaf)

Section shape	Stiffness ϕ_B^e	Stiffness ϕ_T^e	Strength ϕ_B^f	Strength ϕ_T^f
	$\frac{\pi}{6}\frac{b}{t}$	$\frac{\pi}{8}\frac{b}{t}\left(1-\frac{t}{b}\right)^4$	$\frac{2}{3}\sqrt{\pi}\left(\frac{b}{t}\right)^{1/2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{b}{t}\right)^{1/2}\left(1-\frac{t}{b}\right)^2$
	$\frac{a}{t}\frac{(1+3b/a)}{(1+b/a)^2}$	$\frac{8(ab)^{5/2}}{t(a^2+b^2)(a+b)^2}$	$\left(\frac{a}{t}\right)^{1/2}\frac{(1+3b/a)}{(1+b/a)^{3/2}}$	$\frac{4a^{1/2}}{t^{1/2}(1+a/b)^{3/2}}$
	$\frac{\pi}{2}\frac{h^2}{bt}$	—	$\sqrt{2\pi}\frac{h}{(bt)^{1/2}}$	—
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+3b/h)}{(1+b/h)^2}$	 $\frac{\pi b^2 h^2}{t(h+b)^3}$  $\frac{\pi}{3}\frac{t}{b}\frac{(1+4h/b)}{(1+h/b)^2}$	$\frac{\sqrt{2\pi}}{3}\left(\frac{h}{t}\right)^{1/2}\frac{(1+3b/h)}{(1+b/h)^{3/2}}$	 $\frac{\sqrt{2\pi}h}{(bt)^{1/2}(1+h/b)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{b}\right)^{1/2}\frac{(1+4h/b)}{(1+h/b)^{3/2}}$
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+4bt^2/h^3)}{(1+b/h)^2}$	 $\frac{\pi}{6}\frac{t}{h}\frac{(1+8b/h)}{(1+b/h)^2}$  $\frac{\pi}{3}\frac{t}{h}\frac{(1+4b/h)}{(1+b/h)^2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{h}{t}\right)^{1/2}\frac{(1+4bt^2/h^3)}{(1+b/h)^{3/2}}$	 $\left(\frac{\pi}{18}\frac{t}{h}\right)^{1/2}\frac{(1+8b/h)}{(1+b/h)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{h}\right)^{1/2}\frac{(1+4b/h)}{(1+b/h)^{3/2}}$
	$\frac{\pi}{2}\frac{d^2}{t\lambda}$	—	$\sqrt{\pi}\frac{d}{(t\lambda)^{1/2}}$	—

- Solid equi-axed sections (circles, squares, hexagons, octagons) all have shape factors ϕ_B^e close to 1.

- Elongated sections, I-sections, box-sections etc. can have dramatically different values of ϕ_B^e. For example, a thin-walled tube or a slender I-beam can have a value of ϕ_B^e of 50 or more!

- A beam with $\phi_B^e = 50$ is 50 times stiffer than a solid circular cross-section beam with the same weight!

Elastic Twisting of a Shaft, ϕ_T^e

The **torsional stiffness** of a shaft of length l, made from a material with shear modulus G is

$$S_T = \frac{T}{\theta} = \frac{GK}{l}.$$

The shape enters through the **torsional moment of area** K. For circular sections

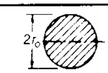
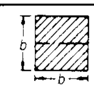
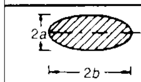
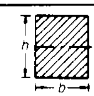
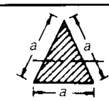
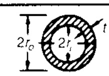
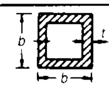
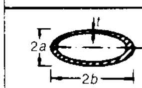
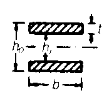
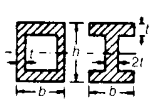
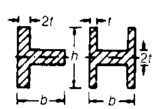
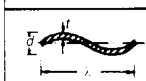
$$K \equiv J = \int_A r^2 \mathrm{d}A,$$

where J is the **polar moment of area**. For non-circular sections

$$K < J$$

Approximate values of K are liste in the attached table.

Moments of areas of sections for common shapes

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}r^3$	$\frac{\pi}{2}r^3$
	b^2	$\frac{b^4}{12}$	$0.14b^4$	$\frac{b^3}{6}$	$0.21b^3$
	πab	$\frac{\pi}{4}a^3b$	$\frac{\pi a^3b^3}{(a^2+b^2)}$	$\frac{\pi}{4}a^2b$	$\frac{\pi a^2b}{2}$ $(a < b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3h}{3}\left(1-0.58\frac{b}{h}\right)$ $(h > b)$	$\frac{bh^2}{6}$	$\frac{b^2h^2}{3h+1.8b}$ $(h > b)$
	$\frac{\sqrt{3}}{4}a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2t$
	$4bt$	$\frac{2}{3}b^3t$	$b^3t\left(1-\frac{t}{b}\right)^4$	$\frac{4}{3}b^2t$	$2b^2t\left(1-\frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4}a^3t\left(1+\frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\frac{\pi a^2t}{4}\left(1+\frac{3b}{a}\right)$	$2\pi t(a^3b)^{1/2}$ $(b > a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx \frac{1}{2}bth_o^2$	—	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\frac{1}{6}h^3t\left(1+\frac{3b}{h}\right)$	□: $\approx \frac{2tb^2h^2}{h+b}$; I: $\frac{2}{3}bt^3\left(1+\frac{4h}{b}\right)$	$\frac{h^2t}{3}\left(1+\frac{3b}{h}\right)$	□: $2tbh$; I: $\frac{2}{3}bt^2\left(1+\frac{4h}{b}\right)$
	$2t(h+b)$	$\frac{t}{6}(h^3+4bt^2)$	⊢: $\frac{t^3}{3}(8b+h)$; H: $\frac{2}{3}ht^3\left(1+\frac{4b}{h}\right)$	$\frac{t}{3h}(h^3+4bt^2)$	⊢: $\frac{t^2}{3}(8b+h)$; H: $\frac{2}{3}ht^2\left(1+\frac{4b}{h}\right)$
	$t\lambda\left(1+\frac{\pi^2d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

Torsional stiffness of a shaft:

$$S_T = \frac{GK}{l}.$$

Let

$$S_T^o = \frac{GK^o}{l}$$

denote the stiffness of the shaft of the same length l, made from the same material with shear modulus G, but with a **solid circular cross-section** with an polar moment of area

$$K^o = \frac{\pi r^4}{2} = \frac{A^2}{2\pi}$$

such that the cross-sectional area A of the shaft with the solid circular section is the same as the area of the shaft with a more general cross-section.

Then the shape factor for elastic twisting of shafts is defined as

$$\phi_T^e \equiv \frac{S_T}{S_T^o} \quad \Rightarrow \quad \phi_T^e = \frac{K}{K^o} = \frac{2\pi K}{A^2}$$

or

$$\boxed{\phi_T^e = \frac{2\pi K}{A^2} \quad \text{dimensionless}}$$

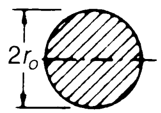
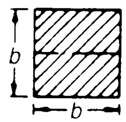
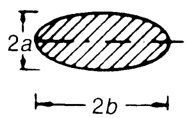
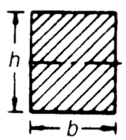
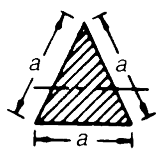
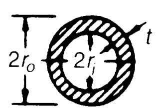
As an example consider a thin-walled tube of mean radius r and wall thickness t. For this cross-section $K = 2\pi r^3 t$ and $A = 2\pi r t$. Hence

$$\phi_T^e = \frac{2\pi 2\pi r^3 t}{(2\pi r t)^2},$$

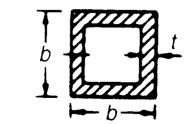
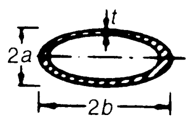
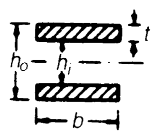
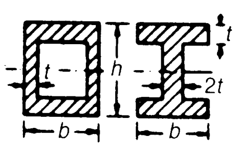
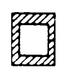
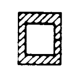
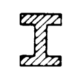
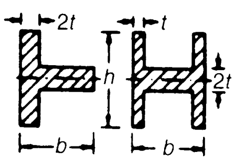
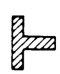
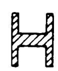
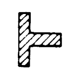
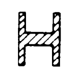
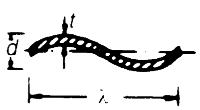
or for a shaft with a thin-walled circular cross-section

$$\boxed{\phi_T^e = \frac{r}{t}}$$

Values for the four shape factors

Section shape	Stiffness ϕ_B^e	Stiffness ϕ_T^e	Strength ϕ_B^f	Strength ϕ_T^f
	1	1	1	1
	$\frac{\pi}{3} = 1.05$	0.88	$\frac{2}{3}\sqrt{\pi} = 1.18$	0.74
	$\frac{a}{b}$	$\frac{2ab}{(a^2 + b^2)}$	$\sqrt{\frac{a}{b}}$	$\sqrt{\frac{a}{b}}$ $(a < b)$
	$\frac{\pi}{3}\frac{h}{b}$	$\frac{2\pi}{3}\frac{b}{h}\left(1 - 0.58\frac{h}{b}\right)$ $(h > b)$	$\frac{2}{3}\sqrt{\pi}\left(\frac{h}{b}\right)^{1/2}$	$\frac{2}{3}\sqrt{\pi}\frac{(b/h)^{1/2}}{(1 + 0.6b/h)}$ $(h > b)$
	$\frac{2\pi}{3\sqrt{3}} = 1.21$	$\frac{2\pi}{5\sqrt{3}} = 0.73$	0.77	0.62
	$\frac{r}{t}$	$\frac{r}{t}$	$\left(\frac{2r}{t}\right)^{1/2}$	$\left(\frac{2r}{t}\right)^{1/2}$

(*continued overleaf*)

Section shape	Stiffness ϕ_B^e	Stiffness ϕ_T^e	Srength ϕ_B^f	Srength ϕ_T^f
	$\frac{\pi}{6}\frac{b}{t}$	$\frac{\pi}{8}\frac{b}{t}\left(1-\frac{t}{b}\right)^4$	$\frac{2}{3}\sqrt{\pi}\left(\frac{b}{t}\right)^{1/2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{b}{t}\right)^{1/2}\left(1-\frac{t}{b}\right)^2$
	$\frac{a}{t}\frac{(1+3b/a)}{(1+b/a)^2}$	$\frac{8(ab)^{5/2}}{t(a^2+b^2)(a+b)^2}$	$\left(\frac{a}{t}\right)^{1/2}\frac{(1+3b/a)}{(1+b/a)^{3/2}}$	$\frac{4a^{1/2}}{t^{1/2}(1+a/b)^{3/2}}$
	$\frac{\pi}{2}\frac{h^2}{bt}$	—	$\sqrt{2\pi}\frac{h}{(bt)^{1/2}}$	—
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+3b/h)}{(1+b/h)^2}$	 $\frac{\pi b^2 h^2}{t(h+b)^3}$  $\frac{\pi}{3}\frac{t}{b}\frac{(1+4h/b)}{(1+h/b)^2}$	$\frac{\sqrt{2\pi}}{3}\left(\frac{h}{t}\right)^{1/2}\frac{(1+3b/h)}{(1+b/h)^{3/2}}$	 $\frac{\sqrt{2\pi}h}{(bt)^{1/2}(1+h/b)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{b}\right)^{1/2}\frac{(1+4h/b)}{(1+h/b)^{3/2}}$
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+4bt^2/h^3)}{(1+b/h)^2}$	 $\frac{\pi}{6}\frac{t}{h}\frac{(1+8b/h)}{(1+b/h)^2}$  $\frac{\pi}{3}\frac{t}{h}\frac{(1+4b/h)}{(1+b/h)^2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{h}{t}\right)^{1/2}\frac{(1+4bt^2/h^3)}{(1+b/h)^{3/2}}$	 $\left(\frac{\pi}{18}\frac{t}{h}\right)^{1/2}\frac{(1+8b/h)}{(1+b/h)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{h}\right)^{1/2}\frac{(1+4b/h)}{(1+b/h)^{3/2}}$
	$\frac{\pi}{2}\frac{d^2}{t\lambda}$	—	$\sqrt{\pi}\frac{d}{(t\lambda)^{1/2}}$	—

Failure in Bending, ϕ_B^f

Under a bending moment M, the stress is the largest in the fibers which lie at the furthest distance, y_m, from the neutral axis of the beam:

$$|\sigma_{max}| = \frac{M y_m}{I} = \frac{M}{Z},$$

where

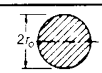
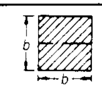
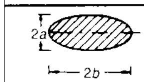
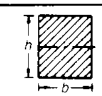
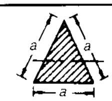
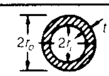
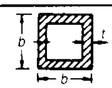
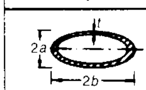
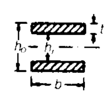
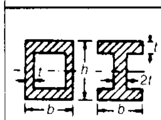
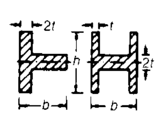
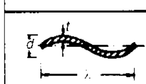
$$Z \equiv \frac{I}{y_m}.$$

is the **section modulus** of the beam. Failure occurs when $|\sigma_{max}|$ reaches σ_f, or when the moment reaches the value

$$M_f = Z\,\sigma_f.$$

The shape enters through the **section modulus** Z.

Moments of areas of sections for common shapes

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}r^3$	$\frac{\pi}{2}r^3$
	b^2	$\frac{b^4}{12}$	$0.14b^4$	$\frac{b^3}{6}$	$0.21b^3$
	πab	$\frac{\pi}{4}a^3b$	$\frac{\pi a^3b^3}{(a^2+b^2)}$	$\frac{\pi}{4}a^2b$	$\frac{\pi a^2b}{2}$ $(a < b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3h}{3}\left(1-0.58\frac{b}{h}\right)$ $(h > b)$	$\frac{bh^2}{6}$	$\frac{b^2h^2}{3h+1.8b}$ $(h > b)$
	$\frac{\sqrt{3}}{4}a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2t$
	$4bt$	$\frac{2}{3}b^3t$	$b^3t\left(1-\frac{t}{b}\right)^4$	$\frac{4}{3}b^2t$	$2b^2t\left(1-\frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4}a^3t\left(1+\frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\frac{\pi a^2t}{4}\left(1+\frac{3b}{a}\right)$	$2\pi t(a^3b)^{1/2}$ $(b > a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx \frac{1}{2}bth_o^2$	—	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\frac{1}{6}h^3t\left(1+\frac{3b}{h}\right)$	□: $\approx\frac{2tb^2h^2}{h+b}$; I: $\frac{2}{3}bt^3\left(1+\frac{4h}{b}\right)$	$\frac{h^2t}{3}\left(1+\frac{3b}{h}\right)$	□: $2tbh$; I: $\frac{2}{3}bt^2\left(1+\frac{4h}{b}\right)$
	$2t(h+b)$	$\frac{t}{6}(h^3+4bt^2)$	H: $\frac{t^3}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^3\left(1+\frac{4b}{h}\right)$	$\frac{t}{3h}(h^3+4bt^2)$	H: $\frac{t^2}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^2\left(1+\frac{4b}{h}\right)$
	$t\lambda\left(1+\frac{\pi^2d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

Failure moment for a shaft in bending:

$$M_f = Z\sigma_f.$$

Let

$$M_f^o = Z^o \sigma_f$$

denote the failure moment of a beam made from the same material with a strength σ_f, but with a **solid circular cross-section** with a section modulus

$$Z^o = \frac{\pi r^3}{4} = \frac{A^{3/2}}{4\sqrt{\pi}}$$

such that the cross-sectional area A of the beam with the solid circular section is the same as the area of the beam with a more general cross-section.

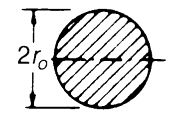
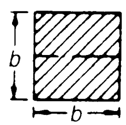
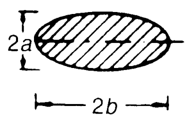
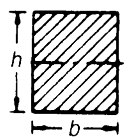
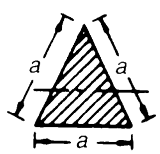
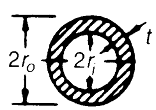
Then the shape factor for failure in bending of beams is defined as

$$\phi_B^f \equiv \frac{M_f}{M_f^o} \quad \Rightarrow \quad \phi_B^f = \frac{Z}{Z^o} = \frac{4\sqrt{\pi}Z}{A^{3/2}}$$

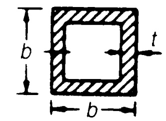
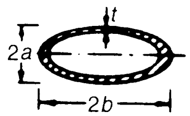
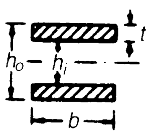
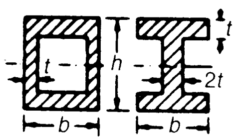
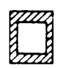
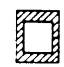
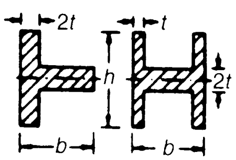
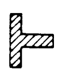
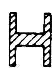
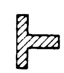
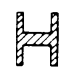
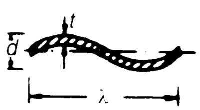
or

$$\boxed{\phi_B^f = \frac{4\sqrt{\pi}Z}{A^{3/2}} \quad \text{dimensionless}}$$

Values for the four shape factors

Section shape	Stiffness ϕ_B^e	ϕ_T^e	Strength ϕ_B^f	ϕ_T^f
	1	1	1	1
	$\frac{\pi}{3} = 1.05$	0.88	$\frac{2}{3}\sqrt{\pi} = 1.18$	0.74
	$\frac{a}{b}$	$\frac{2ab}{(a^2 + b^2)}$	$\sqrt{\frac{a}{b}}$	$\sqrt{\frac{a}{b}}$ $(a < b)$
	$\frac{\pi}{3}\frac{h}{b}$	$\frac{2\pi}{3}\frac{b}{h}\left(1 - 0.58\frac{h}{b}\right)$ $(h > b)$	$\frac{2}{3}\sqrt{\pi}\left(\frac{h}{b}\right)^{1/2}$	$\frac{2}{3}\sqrt{\pi}\frac{(b/h)^{1/2}}{(1 + 0.6b/h)}$ $(h > b)$
	$\frac{2\pi}{3\sqrt{3}} = 1.21$	$\frac{2\pi}{5\sqrt{3}} = 0.73$	0.77	0.62
	$\frac{r}{t}$	$\frac{r}{t}$	$\left(\frac{2r}{t}\right)^{1/2}$	$\left(\frac{2r}{t}\right)^{1/2}$

(continued overleaf)

Section shape	*Stiffness* ϕ_B^e	*Stiffness* ϕ_T^e	*Srength* ϕ_B^f	*Srength* ϕ_T^f
	$\frac{\pi}{6}\frac{b}{t}$	$\frac{\pi}{8}\frac{b}{t}\left(1-\frac{t}{b}\right)^4$	$\frac{2}{3}\sqrt{\pi}\left(\frac{b}{t}\right)^{1/2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{b}{t}\right)^{1/2}\left(1-\frac{t}{b}\right)^2$
	$\frac{a}{t}\frac{(1+3b/a)}{(1+b/a)^2}$	$\frac{8(ab)^{5/2}}{t(a^2+b^2)(a+b)^2}$	$\left(\frac{a}{t}\right)^{1/2}\frac{(1+3b/a)}{(1+b/a)^{3/2}}$	$\frac{4a^{1/2}}{t^{1/2}(1+a/b)^{3/2}}$
	$\frac{\pi}{2}\frac{h^2}{bt}$	—	$\sqrt{2\pi}\frac{h}{(bt)^{1/2}}$	—
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+3b/h)}{(1+b/h)^2}$	 $\frac{\pi b^2h^2}{t(h+b)^3}$  $\frac{\pi}{3}\frac{t}{b}\frac{(1+4h/b)}{(1+h/b)^2}$	$\frac{\sqrt{2\pi}}{3}\left(\frac{h}{t}\right)^{1/2}\frac{(1+3b/h)}{(1+b/h)^{3/2}}$	 $\frac{\sqrt{2\pi}h}{(bt)^{1/2}(1+h/b)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{b}\right)^{1/2}\frac{(1+4h/b)}{(1+h/b)^{3/2}}$
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+4bt^2/h^3)}{(1+b/h)^2}$	 $\frac{\pi}{6}\frac{t}{h}\frac{(1+8b/h)}{(1+b/h)^2}$  $\frac{\pi}{3}\frac{t}{h}\frac{(1+4b/h)}{(1+b/h)^2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{h}{t}\right)^{1/2}\frac{(1+4bt^2/h^3)}{(1+b/h)^{3/2}}$	 $\left(\frac{\pi}{18}\frac{t}{h}\right)^{1/2}\frac{(1+8b/h)}{(1+b/h)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{h}\right)^{1/2}\frac{(1+4b/h)}{(1+b/h)^{3/2}}$
	$\frac{\pi}{2}\frac{d^2}{t\lambda}$	—	$\sqrt{\pi}\frac{d}{(t\lambda)^{1/2}}$	—

Failure in Twisting, ϕ_T^f

For a solid circular shaft under a twisting moment T, the shear stress is the largest at the furthest distance, r_m, from the axis of the shaft:

$$\tau_{max} = \frac{T r_m}{J}.$$

For non-circular sections with ends that are free to warp, the maximum shear stress on the surface of the shaft is given by

$$\tau_{max} = \frac{T}{Q},$$

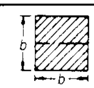
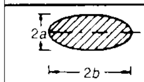
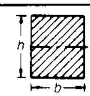
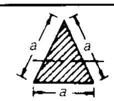
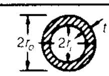
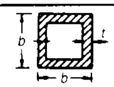
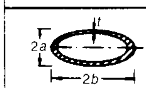
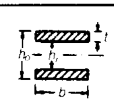
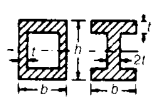
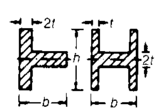
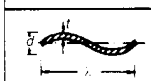
where Q, with units of m^3 plays the same role as J/r_m.

Failure occurs when τ_{max} reaches $\approx \sigma_f/2$, or when the twisting moment reaches the value

$$T_f = Q\,(\sigma_f/2).$$

The shape enters through the factor Q.

Moments of areas of sections for common shapes

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4} r^4$	$\frac{\pi}{2} r^4$	$\frac{\pi}{4} r^3$	$\frac{\pi}{2} r^3$
	b^2	$\frac{b^4}{12}$	$0.14b^4$	$\frac{b^3}{6}$	$0.21b^3$
	πab	$\frac{\pi}{4} a^3 b$	$\frac{\pi a^3 b^3}{(a^2+b^2)}$	$\frac{\pi}{4} a^2 b$	$\frac{\pi a^2 b}{2}$ $(a < b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3 h}{3}\left(1 - 0.58\frac{b}{h}\right)$ $(h > b)$	$\frac{bh^2}{6}$	$\frac{b^2 h^2}{3h + 1.8b}$ $(h > b)$
	$\frac{\sqrt{3}}{4} a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2 - r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4 - r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2}(r_o^4 - r_i^4)$ $\approx 2\pi r^3 t$	$\frac{\pi}{4r_o}(r_o^4 - r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{2r_o}(r_o^4 - r_i^4)$ $\approx 2\pi r^2 t$
	$4bt$	$\frac{2}{3} b^3 t$	$b^3 t\left(1 - \frac{t}{b}\right)^4$	$\frac{4}{3} b^2 t$	$2b^2 t\left(1 - \frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4} a^3 t\left(1 + \frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2} t}{(a^2+b^2)}$	$\frac{\pi a^2 t}{4}\left(1 + \frac{3b}{a}\right)$	$2\pi t(a^3 b)^{1/2}$ $(b > a)$
	$b(h_o - h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3 - h_i^3)$ $\approx \frac{1}{2} bth_o^2$	—	$\frac{b}{6h_o}(h_o^3 - h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\frac{1}{6} h^3 t\left(1 + \frac{3b}{h}\right)$	□: $\approx \frac{2tb^2h^2}{h+b}$ / I: $\frac{2}{3} bt^3\left(1 + \frac{4h}{b}\right)$	$\frac{h^2 t}{3}\left(1 + \frac{3b}{h}\right)$	□: $2tbh$ / I: $\frac{2}{3} bt^2\left(1 + \frac{4h}{b}\right)$
	$2t(h+b)$	$\frac{t}{6}(h^3 + 4bt^2)$	⊢: $\frac{t^3}{3}(8b+h)$ / H: $\frac{2}{3} ht^3\left(1 + \frac{4b}{h}\right)$	$\frac{t}{3h}(h^3 + 4bt^2)$	⊢: $\frac{t^2}{3}(8b+h)$ / H: $\frac{2}{3} ht^2\left(1 + \frac{4b}{h}\right)$
	$t\lambda\left(1 + \frac{\pi^2 d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

Failure moment for twisting of a shaft:

$$T_f = Q(\sigma_f/2).$$

Let

$$T_f^o = Q^o(\sigma_f/2)$$

denote the failure moment of a beam made from the same material with a strength σ_f, but with a **solid circular cross-section** with

$$Q^o = \frac{\pi r^3}{2} = \frac{A^{3/2}}{2\sqrt{\pi}}$$

such that the cross-sectional area A of the beam with the solid circular section is the same as the area of the beam with a more general cross-section.

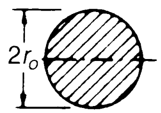
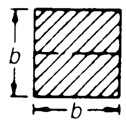
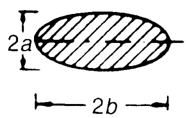
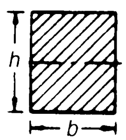
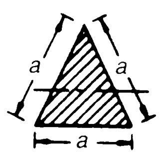
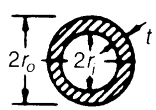
Then the shape factor for failure in bending of beams is defined as

$$\phi_T^f \equiv \frac{T_f}{T_f^o} \quad \Rightarrow \quad \phi_T^f = \frac{T}{T^o} = \frac{2\sqrt{\pi}Q}{A^{3/2}}$$

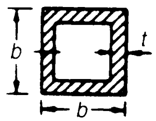
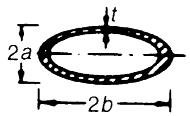
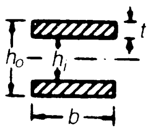
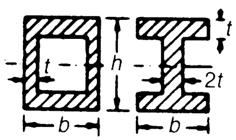
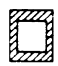
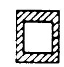
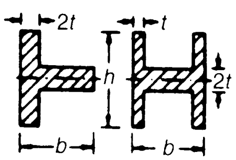
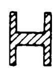
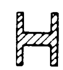
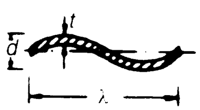
or

$$\boxed{\phi_T^f = \frac{2\sqrt{\pi}Q}{A^{3/2}} \quad \text{dimensionless}}$$

Values for the four shape factors

Section shape	Stiffness ϕ_B^e	ϕ_T^e	Strength ϕ_B^f	ϕ_T^f
	1	1	1	1
	$\frac{\pi}{3} = 1.05$	0.88	$\frac{2}{3}\sqrt{\pi} = 1.18$	0.74
	$\frac{a}{b}$	$\frac{2ab}{(a^2+b^2)}$	$\sqrt{\frac{a}{b}}$	$\sqrt{\frac{a}{b}}$ $(a < b)$
	$\frac{\pi}{3}\frac{h}{b}$	$\frac{2\pi}{3}\frac{b}{h}\left(1 - 0.58\frac{h}{b}\right)$ $(h > b)$	$\frac{2}{3}\sqrt{\pi}\left(\frac{h}{b}\right)^{1/2}$	$\frac{2}{3}\sqrt{\pi}\frac{(b/h)^{1/2}}{(1+0.6b/h)}$ $(h > b)$
	$\frac{2\pi}{3\sqrt{3}} = 1.21$	$\frac{2\pi}{5\sqrt{3}} = 0.73$	0.77	0.62
	$\frac{r}{t}$	$\frac{r}{t}$	$\left(\frac{2r}{t}\right)^{1/2}$	$\left(\frac{2r}{t}\right)^{1/2}$

(continued overleaf)

Section shape	Stiffness ϕ_B^e	Stiffness ϕ_T^e	Srength ϕ_B^f	Srength ϕ_T^f
	$\frac{\pi}{6}\frac{b}{t}$	$\frac{\pi}{8}\frac{b}{t}\left(1-\frac{t}{b}\right)^4$	$\frac{2}{3}\sqrt{\pi}\left(\frac{b}{t}\right)^{1/2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{b}{t}\right)^{1/2}\left(1-\frac{t}{b}\right)^2$
	$\frac{a}{t}\frac{(1+3b/a)}{(1+b/a)^2}$	$\frac{8(ab)^{5/2}}{t(a^2+b^2)(a+b)^2}$	$\left(\frac{a}{t}\right)^{1/2}\frac{(1+3b/a)}{(1+b/a)^{3/2}}$	$\frac{4a^{1/2}}{t^{1/2}(1+a/b)^{3/2}}$
	$\frac{\pi}{2}\frac{h^2}{bt}$	—	$\sqrt{2\pi}\frac{h}{(bt)^{1/2}}$	—
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+3b/h)}{(1+b/h)^2}$	 $\frac{\pi b^2 h^2}{t(h+b)^3}$  $\frac{\pi}{3}\frac{t}{b}\frac{(1+4h/b)}{(1+h/b)^2}$	$\frac{\sqrt{2\pi}}{3}\left(\frac{h}{t}\right)^{1/2}\frac{(1+3b/h)}{(1+b/h)^{3/2}}$	 $\frac{\sqrt{2\pi}h}{(bt)^{1/2}(1+h/b)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{b}\right)^{1/2}\frac{(1+4h/b)}{(1+h/b)^{3/2}}$
	$\frac{\pi}{6}\frac{h}{t}\frac{(1+4bt^2/h^3)}{(1+b/h)^2}$	 $\frac{\pi}{6}\frac{t}{h}\frac{(1+8b/h)}{(1+b/h)^2}$  $\frac{\pi}{3}\frac{t}{h}\frac{(1+4b/h)}{(1+b/h)^2}$	$\frac{\sqrt{\pi}}{2}\left(\frac{h}{t}\right)^{1/2}\frac{(1+4bt^2/h^3)}{(1+b/h)^{3/2}}$	 $\left(\frac{\pi}{18}\frac{t}{h}\right)^{1/2}\frac{(1+8b/h)}{(1+b/h)^{3/2}}$  $\frac{\sqrt{2\pi}}{3}\left(\frac{t}{h}\right)^{1/2}\frac{(1+4b/h)}{(1+b/h)^{3/2}}$
	$\frac{\pi}{2}\frac{d^2}{t\lambda}$	—	$\sqrt{\pi}\frac{d}{(t\lambda)^{1/2}}$	—

Column buckling under compressive axial loading

Euler buckling load:

$$F_cr = c\frac{\pi^2 EI_{min}}{l^2}.$$

The resistance to bukling depends on the smallest second moment of arae I_{min}.

The appropriate shape factor is then the same as that for elastic bending with I_{min} replacing I:

$$\phi_B^e = \frac{4\pi I_{min}}{A^2}.$$

- A beam or a shaft with an elastic shape factor of 50 is fifty times stiffer than a solid circular section of the same mass per unit length.

- A beam or a shaft with a failure shape factor of 20 is twenty times stonger than a solid circular section of the same mass per unit length.

- If one wishes to make stiff, strong structures which are efficient, that is those which as little material as possible, then making the shape factors as large as possible is the way to achieve this goal.

- The bigger the value of the ϕ the better!

- However, there are limits to the maximum values of ϕ!

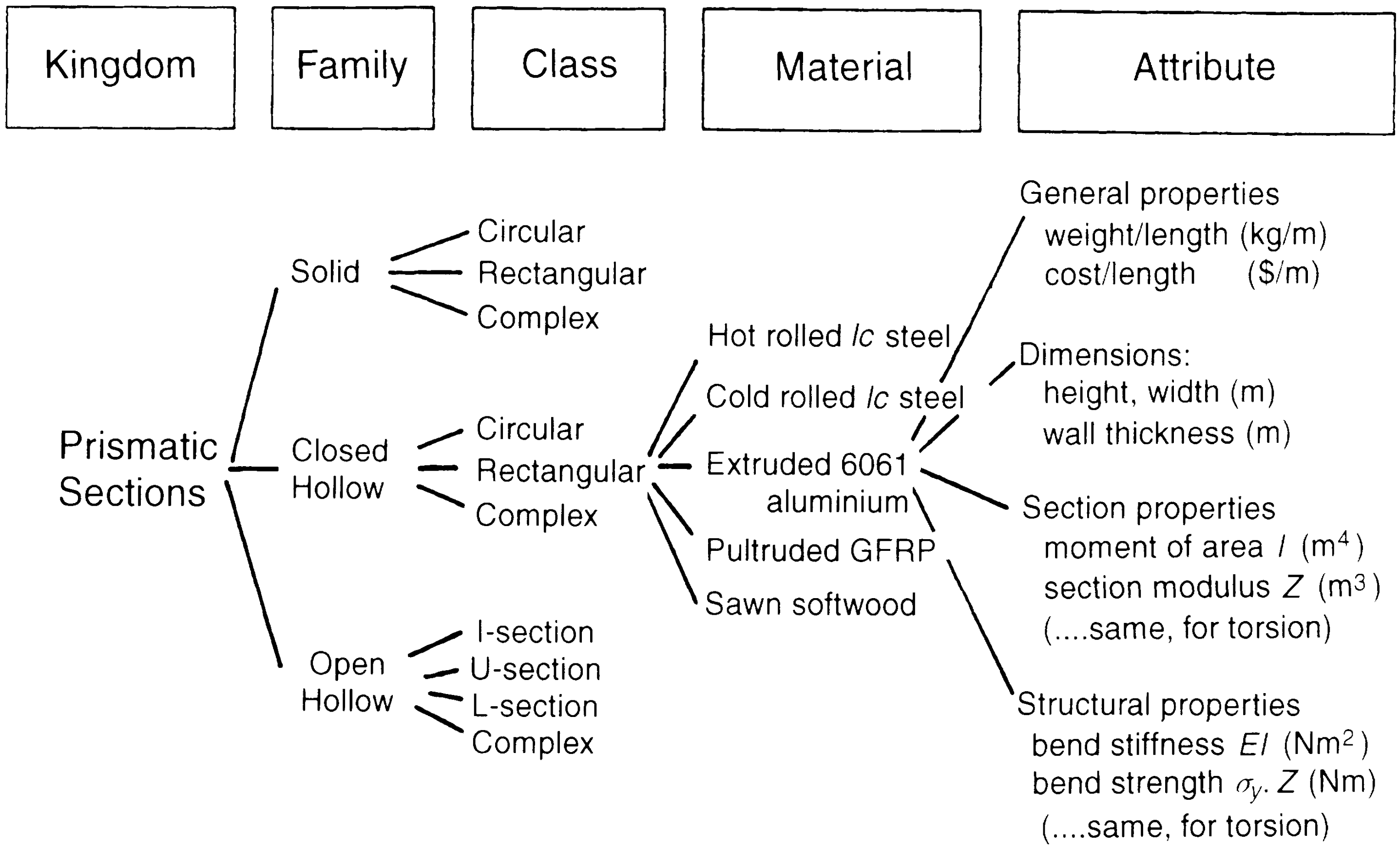
- The limits on the maximum values of ϕ are imposed by the limits on the thinness of sections.

- These limits may be imposed by

  - Manufacturing constraints: the difficulty or expense of making an efficient shape may be too high.

  - Mechanical stability of thin sections — local buckling!.

Efficiency Of Standard Sections



A taxonomy of prismatic shapes, illustrating the attributes of a shaped section.

Recall

$$\phi_B^e = \frac{4\pi I}{A^2}, \qquad \phi_T^e = \frac{2\pi K}{A^2}, \qquad \phi_B^f = \frac{4\sqrt{\pi}Z}{A^{3/2}}, \qquad \phi_T^f = \frac{2\sqrt{\pi}Q}{A^{3/2}}.$$

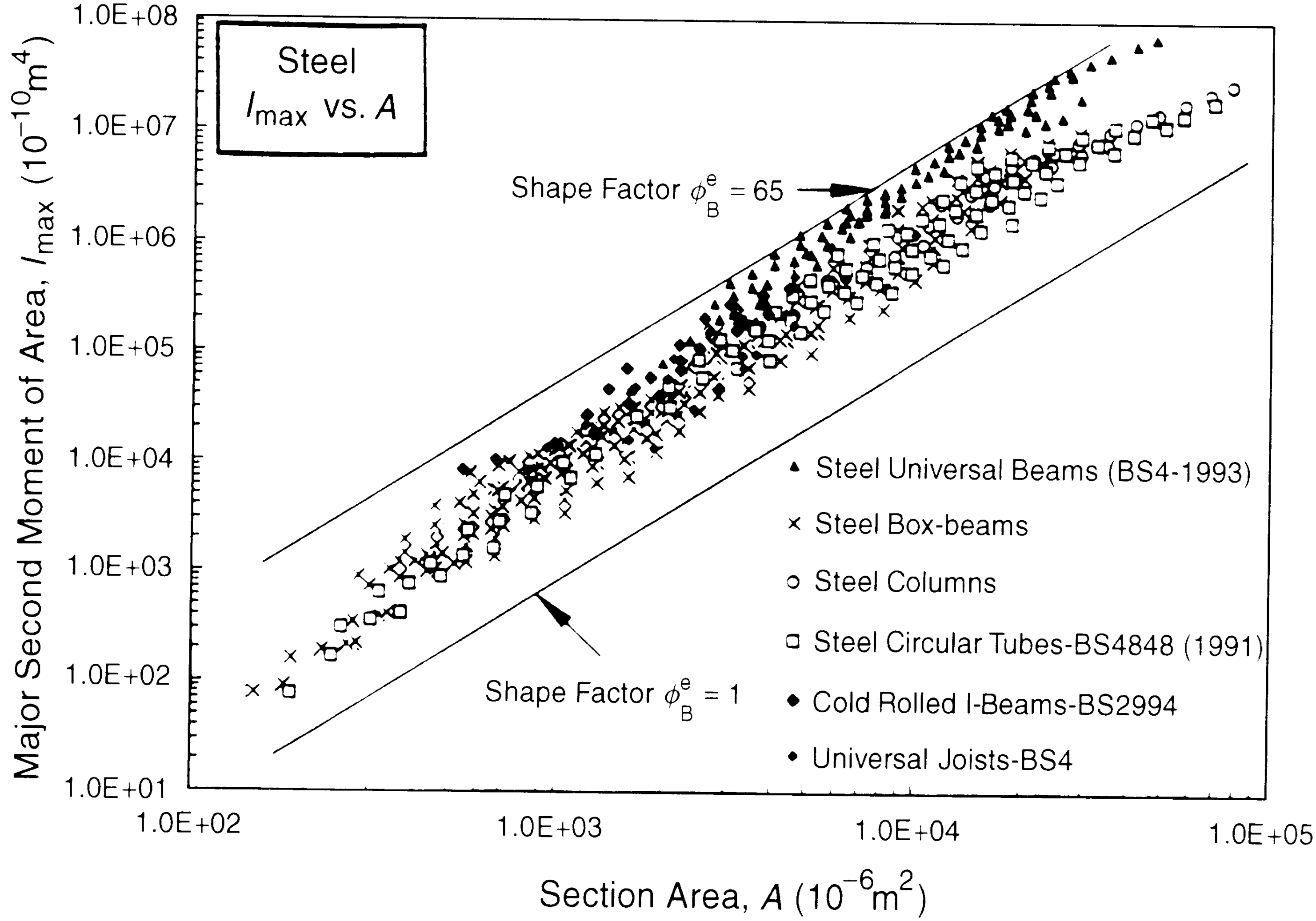
Thus

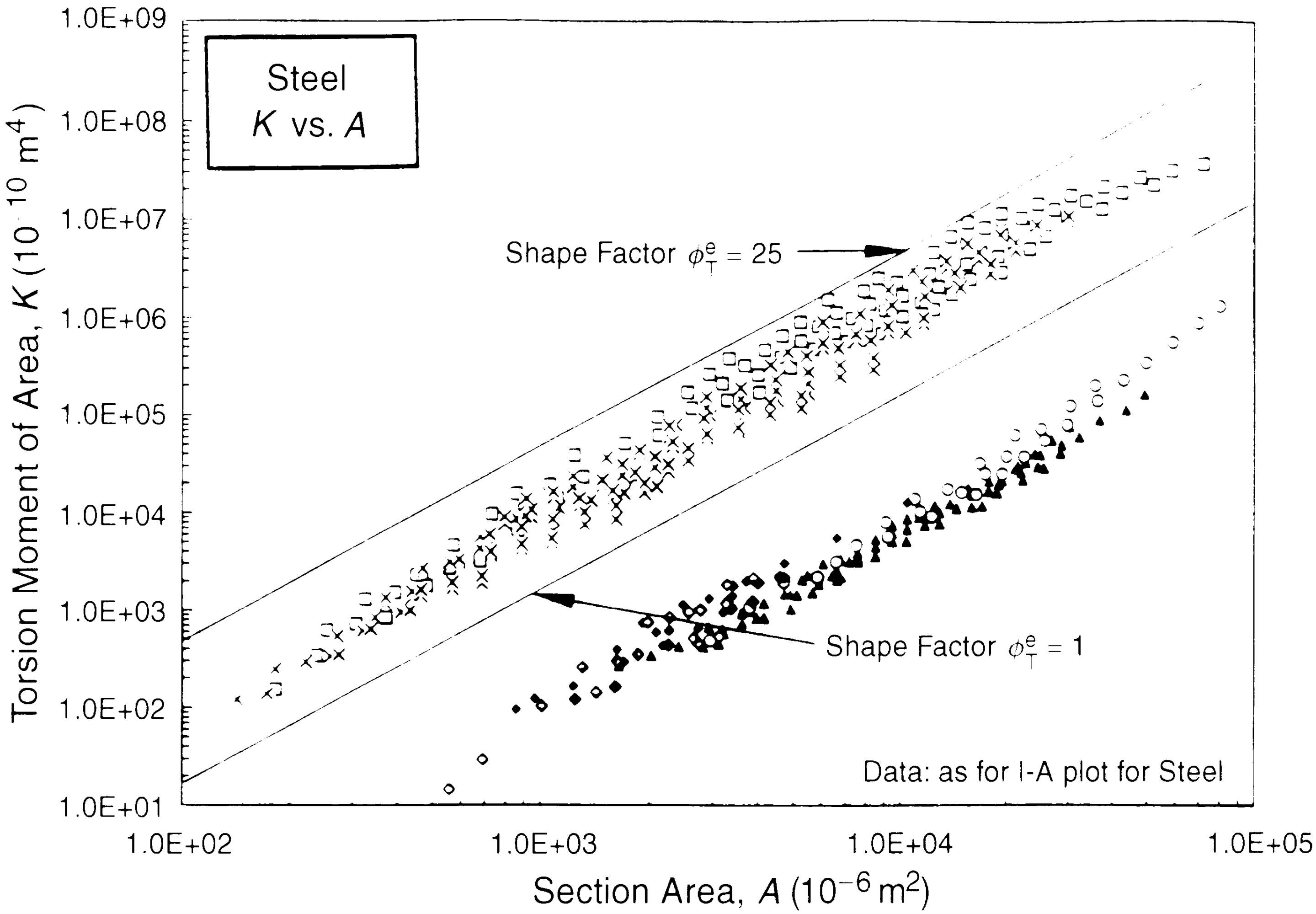
$$\log I = \log \frac{\phi_B^e}{4\pi} + 2 \log A$$

$$\log K = \log \frac{\phi_T^e}{2\pi} + 2 \log A$$

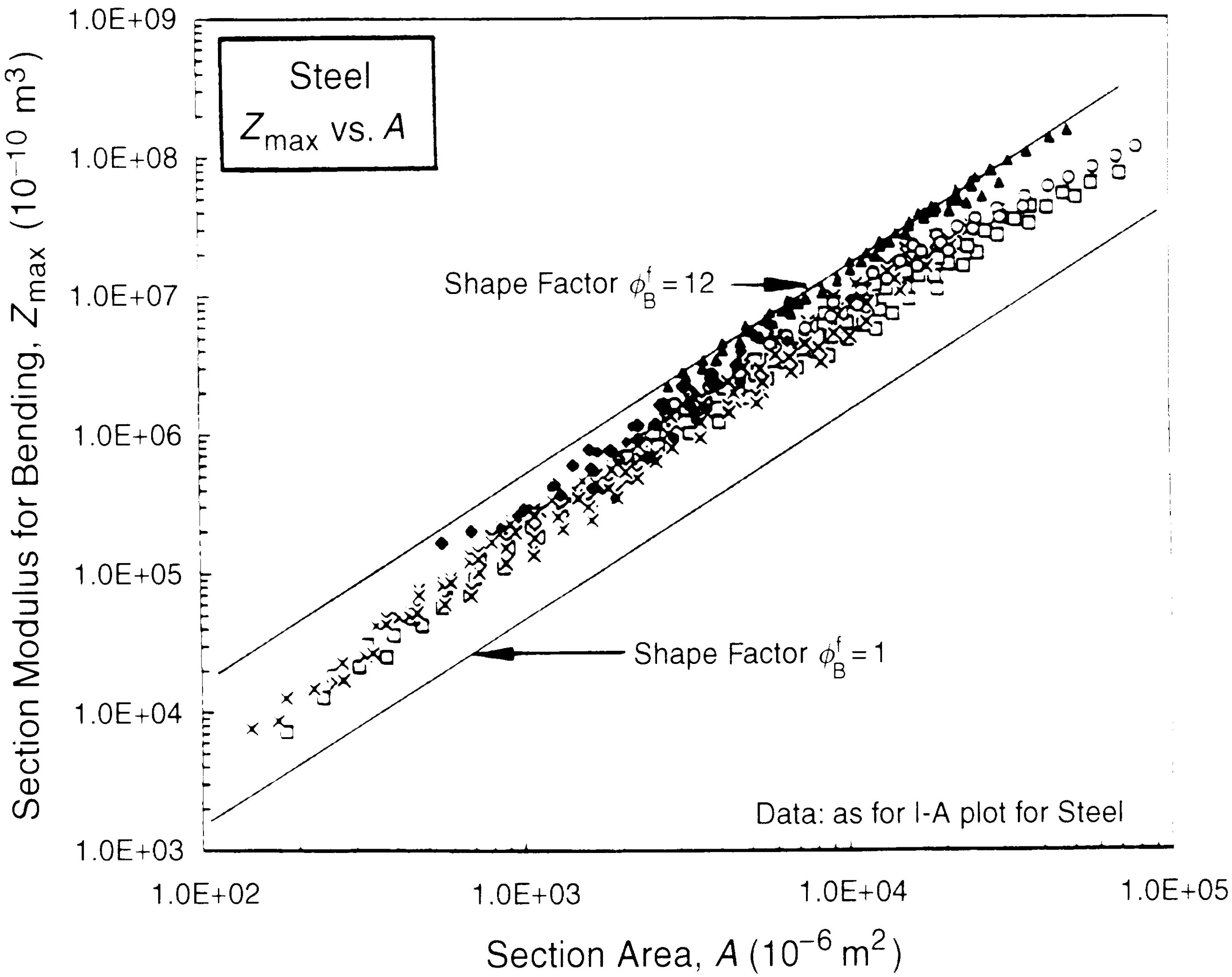
$$\log Z = \log \frac{\phi_B^f}{4\sqrt{\pi}} + \frac{3}{2} \log A$$

$$\log Q = \log \frac{\phi_T^f}{2\sqrt{\pi}} + \frac{3}{2} \log A$$


Steel
I_{max} vs. A
Shape Factor $\phi_B^e = 65$
Shape Factor $\phi_B^e = 1$
Steel Universal Beams (BS4-1993)
Steel Box-beams
Steel Columns
Steel Circular Tubes-BS4848 (1991)
Cold Rolled I-Beams-BS2994
Universal Joists-BS4
Major Second Moment of Area, I_{max} ($10^{-10}m^4$)
1.0E+08
1.0E+07
1.0E+06
1.0E+05
1.0E+04
1.0E+03
1.0E+02
1.0E+01
1.0E+02
1.0E+03
1.0E+04
1.0E+05
Section Area, A ($10^{-6}m^2$)

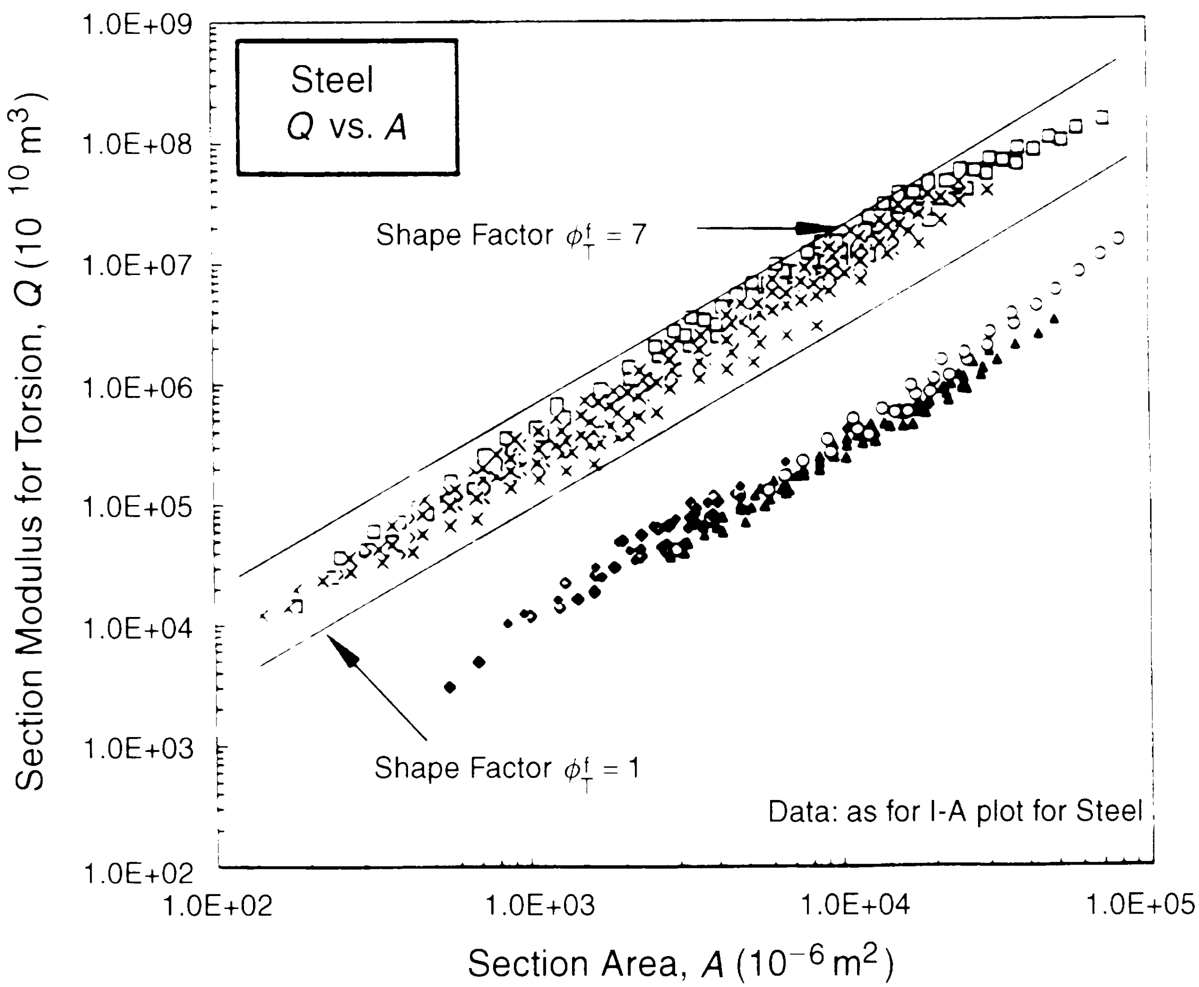


Steel
K vs. A
Shape Factor $\phi_T^e = 25$
Shape Factor $\phi_T^e = 1$
Data: as for I-A plot for Steel
Torsion Moment of Area, K (10^{-10} m^4)
Section Area, A (10^{-6} m^2)
1.0E+01
1.0E+02
1.0E+03
1.0E+04
1.0E+05
1.0E+06
1.0E+07
1.0E+08
1.0E+09



Steel
Z_{max} vs. A
Shape Factor $\phi_B^f = 12$
Shape Factor $\phi_B^f = 1$
Data: as for I-A plot for Steel
Section Modulus for Bending, Z_{max} (10^{-10} m^3)
Section Area, A (10^{-6} m^2)
1.0E+09
1.0E+08
1.0E+07
1.0E+06
1.0E+05
1.0E+04
1.0E+03
1.0E+02
1.0E+03
1.0E+04
1.0E+05

(b)


Steel
Q vs. A
Shape Factor $\phi_T^f = 7$
Shape Factor $\phi_T^f = 1$
Data: as for I-A plot for Steel
Section Modulus for Torsion, Q (10 10 m3)
1.0E+09
1.0E+08
1.0E+07
1.0E+06
1.0E+05
1.0E+04
1.0E+03
1.0E+02
1.0E+02
1.0E+03
1.0E+04
1.0E+05
Section Area, A (10^{-6} m2)

Upper limits for the shape factors ϕ_B^e, ϕ_T^e, ϕ_B^f and ϕ_T^f

Material	$(\phi_B^e)_{max}$	$(\phi_T^e)_{max}$	$(\phi_B^f)_{max}$	$(\phi_T^f)_{max}$
Structural steels	65	25	13	7
Aluminium alloys	44	31	10	8
GFRP and CFRP	39	26	9	7
Polymers (e.g. nylons)	12	8	5	4
Woods (solid sections)	5	1	3	1
Elastomers	<6	3	—	—

- **The upper-limiting shape factors for simple prismatic shapes is governed by local buckling it is material dependent**. For thin-walled tubes

$$\phi_{opt} = \left(\frac{r}{t}\right)_{opt} \propto \frac{E}{\sigma_f} \approx 0.2\frac{E}{\sigma_f}$$

Material Indices Which Include Shape

Select a material for a light stiff beam of length l, to support a bending load F without deflecting too much. That is, the stiffness S is specified. It is to be of minimum mass.

- **Function**: Beam
- **Objective**: Minimize the mass
- **Constraints**: Length l specified. Bending Stiffness S_B specified

Minimize

$$m = A\, l\, \rho \quad \text{Objective function,}$$

subject to

$$\frac{F}{\delta} = \frac{C_1 E I}{l^3} \geq S_B \quad \text{Constraint.}$$

Now

$$\phi_B^e = \frac{4\pi I}{A^2}$$

$$\frac{C_1 E I}{l^3} \geq S_B \;\Rightarrow\; \frac{C_1 E}{l^3}\frac{\phi_B^e}{4\pi}A^2 \geq S_B \;\Rightarrow\; A \geq (4\pi S_B)^{(1/2)} \left(\frac{l^3}{C_1}\right)^{1/2} \frac{1}{(E\phi_B^e)^{1/2}}$$

The smallest cross-sectional area A which meets the stiffness constraint is

$$A = (4\pi S_B)^{(1/2)} \left(\frac{l^3}{C_1}\right)^{1/2} \frac{1}{(E\phi_B^e)^{1/2}}$$

Substituting this in the objective function $m = Al\rho$ gives

$$m = \left\{(4\pi S_B)^{(1/2)}\right\}\left\{\left(\frac{l^5}{C_1}\right)^{1/2}\right\}\left\{\left(\frac{\rho}{(E\phi_B^e)^{1/2}}\right)\right\}$$

Thus the mass will be minimized by selecting the largest value of the index

$$\boxed{M_1 = \frac{\left[E\phi_B^e\right]^{1/2}}{\rho}}$$

Material & Shape Index For Elastic Bending

Exactly the same result holds for elastic buckling of an axially loaded column:

$$\boxed{M_1 = \frac{\left[E\phi_B^e\right]^{1/2}}{\rho}}$$

Material & Shape Index For Elastic Buckling

Select a material for a light stiff shaft of length l, to support a twisting moment T without twisting too much. That is, the stiffness S_T is specified. It is to be of minimum mass.

- **Function**: Shaft

- **Objective**: Minimize the mass

- **Constraints**: Length l specified. Torsional Stiffness S_T specified.

Minimize

$$m = A\, l\, \rho \quad \text{Objective function,}$$

subject to

$$\frac{T}{\theta} = \frac{GK}{l} \geq S_T \quad \text{Constraint.}$$

Now

$$\phi_T^e = \frac{2\pi K}{A^2}$$

$$\frac{GK}{l} \geq S_T \;\Rightarrow\; \frac{G}{l}\frac{\phi_T^e}{2\pi}A^2 \geq S_T \;\Rightarrow\; A \geq (2\pi S_T)^{(1/2)}\, l^{1/2}\frac{1}{(G\phi_T^e)^{1/2}}$$

The smallest cross-sectional area A which meets the stiffness constraint is

$$A = (2\pi S_T)^{(1/2)}\, l^{1/2}\frac{1}{(G\phi_T^e)^{1/2}}$$

Substituting this in the objective function $m = Al\rho$ gives

$$m = \left\{(2\pi S_T)^{(1/2)}\right\}\left\{l^{3/2}\right\}\left\{\left(\frac{\rho}{(G\phi_T^e)^{1/2}}\right)\right\}$$

Thus the mass will be minimized by selecting the largest value of the index

$$\boxed{M_2 = \frac{\left[G\phi_T^e\right]^{1/2}}{\rho}}$$

Material & Shape Index For Elastic Torsion

Failure of Beams and shafts

$$\boxed{M_3 = \frac{\left[\sigma_f \phi_B^f\right]^{2/3}}{\rho}}$$

Material & Shape Index for Failure in Bending

$$\boxed{M_4 = \frac{\left[\sigma_f \phi_T^f\right]^{2/3}}{\rho}}$$

Material & Shape Index for Failure in Torsion

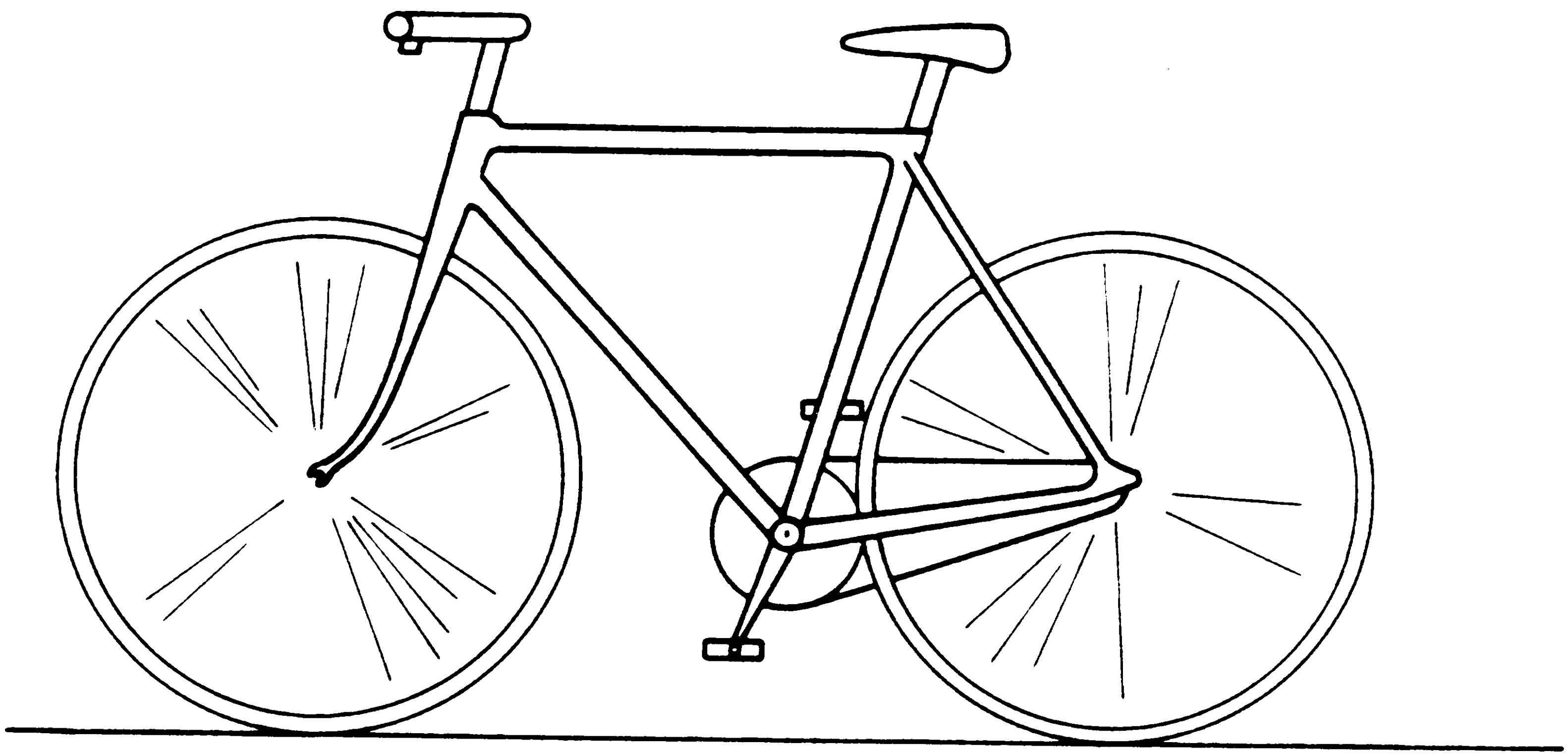
Examples of indices which include shape

(a) Stiffness and strength-limited design at minimum weight (or cost*)

Component shape, loading and constraints	*Stiffness-limited design*	*Strength-limited design*
Tie (tensile member)		
Load, stiffness and length specified, section-area free	$\frac{E}{\rho}$	$\frac{\sigma_f}{\rho}$
Beam (loaded in bending)		
Loaded externally or by self weight, stiffness, strength and length specified, section area free	$\frac{(\phi_B^e E)^{1/2}}{\rho}$	$\frac{(\phi_B^f \sigma_f)^{2/3}}{\rho}$
Torsion bar or **tube**		
Loaded externally , stiffness, strength and length specified, section area free	$\frac{(\phi_T^e E)^{1/2}}{\rho}$	$\frac{(\phi_T^f \sigma_f)^{2/3}}{\rho}$
Column (compression strut)		
Collapse load by buckling or plastic crushing and length specified, section area free	$\frac{(\phi_B^e E)^{1/2}}{\rho}$	$\frac{\sigma_f}{\rho}$

*For cost, replace ρ by $C_m\rho$ in the indices.

EXAMPLE: FORKS FOR A RACING BICYCLE



- The first consideration in bicycle design is strength. Of-course stiffness matters, but the first consideration is that the frames and forks should not yield or fracture in normal use.

- The loading of the forks is predominantly **bending**.

- Since the bicycle is for racing, than mass is a primary consideration. The forks should be as light as possible.

- **Function**: Bicycle forks
- **Objective**: Minimize the mass
- **Constraints**:
  - Must not fail under design loads – strength constraint.
  - Length L specified.

Further details of the load and geometry at this stage are not necessary. The best material and shape is that with the greatest value of

$$\boxed{M_3 = \frac{\left[\sigma_f \phi_B^f\right]^{2/3}}{\rho}}$$

Material & Shape Index for Failure in Bending

Material for bicycle forks

Material	*Strength* σ_f *(MPa)*	*Density* ρ *(Mg/m³)*	*Shape factor* ϕ_B^f	*Index* $\sigma_f^{2/3}/\rho$	*Index* M_3^* $((MPa)^{2/3}/Mg/m^3)$
Spruce (Norwegian)	70–80	0.46–0.56	1–1.5	**36**	36–50
Bamboo	80–160	0.6–0.8	2.4–2.8	(33)	59–65
Steel (Reynolds 531)	770–990	7.82–7.83	7–8	12	44–48
Alu (6061–T6)	240–260	2.69–2.71	5.5–6.3	15	47–51
Titanium 6-4	930–980	4.42–4.43	5.5–6.3	22	69–75
Magnesium AZ 91	160–170	1.80–1.81	4–4.5	17	42–46
CFRP	300–450	1.5–1.6	4–4.5	33	**83–90**

*The range of values of the indices are based on means of the material properties and corresponds to the range of values of ϕ_B^e.

- Solid spruce and bamboo are remarkably efficient; without shape hey are better than any of the others.

- When shape is added to other materials the ranking changes.

- Steel (low alloy 4130-type) and aluminum (6061-T6) are good; Ti-6-4 (sometimes T-3Al-2.5V) is better; CFRP is better still.

- Bicycles have been made from all seven of the materials listed.

- Early bicycles were made from wood; present day racing bicycles are made of steel, aluminum or CFRP. Mountain bicycles have steel or titanium forks.

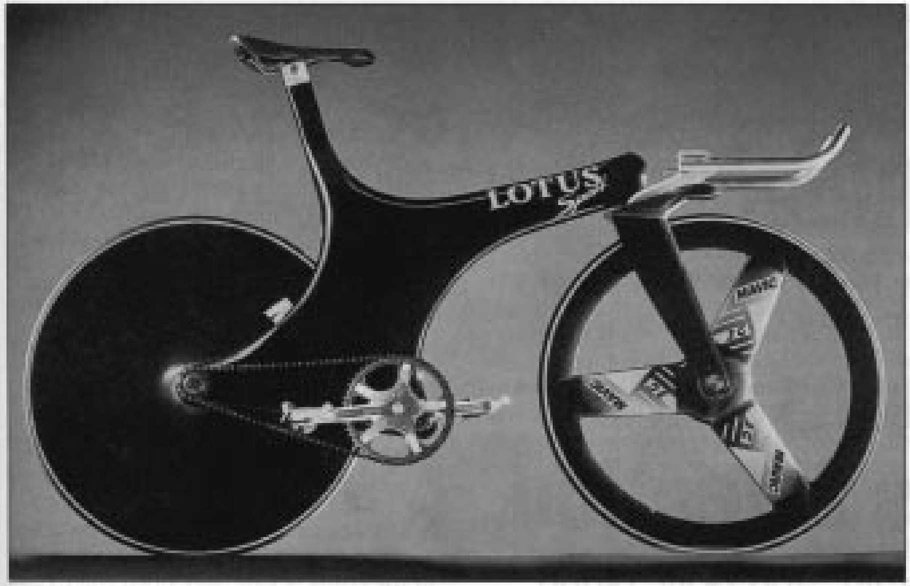


Figure 1. The Lotus monocoque pursuit racing bicycle.

Figure 2. Road bicycle made by Harry Havnoonian from a mix of materials using adhesive bonding for joints, investment-cast stainless steel lugs, Ti-3Al-2.5V head tube, SiC-fiber-reinforced aluminum alloy front triangle, CFRP rear triangle, and CFRP rear wheel.