Introduction to Linear Programming (LP)

Mathematical Programming (MP) Concept

- **Definition**
  - M.P. comprises a range of powerful computer-based optimization methods
- **Approach to Optimization**
  - M.P exploits peculiar features of the structure of a problem to get solutions efficiently
  - Different M.P. methods exploit different structures
- **Two Types**
  - Methods which define optimum -- LP, etc.
    - *Convex feasible regions*
  - Methods which "enumerate" solutions to discover optimum -- Dynamic programming, etc.
    - *Non-convex feasible regions*

LP Concepts

- Special form of mathematical programming
  - Equations must be linear
- Uses simple solution procedures
  - Linear algebra
- Very powerful
  - Extremely large problems
  - 100,000 variables
  - 1000's of constraints

  Note difference!
- Useful design information through Sensitivity Analysis
  - Answers to "what if" questions

Standard Form of LP - Three Parts

- Objective function
  - maximize or minimize

  →
  - $Y = \Sigma c_i x_i$
  - $Y = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

X_i known as decision variables

- Constraints
  - subject to:

  →
  - $a_{11} x_1 + a_{12} x_2 + \ldots <=> b_1$
  - $a_{21} x_1 + a_{22} x_2 + \ldots <=> b_2$
  - $a_{31} x_1 + a_{32} x_2 + \ldots <=> b_3$

- Non-Negativity

  →
  - $x_i \geq 0$ for all i

Standard Form of LP -- Summary

Optimize $Y = \underline{c}\,\underline{X}$

subject to: $\underline{A}\,\underline{X}$ (< or = or >) $\underline{b}$

$X \underline{>} 0$

Three LP Assumptions

1. Linearity
2. Additivity
3. Non-Negativity

- Linearity of Objective Function and Constraints
  - Essential Condition is:

    $f(k\underline{X}) = k\, f(\underline{X})$

  - for example: $f(X) = 3 + 4X_1 + 2X_2$
    is NOT linear in the LP sense
- Implies
  - Constant returns to scale (only first order terms)
  - No "fixed charges" (no constants)

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Three LP Assumptions - continued

- **Additivity:**

  $f(X_1, X_2, \ldots, X_n) = f(X_1) + f(X_2) + \ldots + f(X_n)$

  - no interactive effects among Xi terms
  - assumes that individual segments of the problem operate as well independently as together

- **Non-Negativity**

  $X_i > 0$

  - no fundamental difficulties except in particular situations

Consequences of Assumptions

a. **Convexity of feasible region (if it exists!)**

b. **Convex feasible region, with linear objective function, implies:**
  - Optimum will be on an edge of the feasible region

c. **Since edges are also linear**
  - Optimum is at a corner point (can be several in special cases)

d. **Note that corner points**
  - Constitute small, finite set
  - Defined by solution of linear equations

Bottom Line: Assumptions imply the existence of an efficient solution strategy

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Example for Assumptions

minimize: $Z = 3X_1 + 5X_2$

s.t. $X_1 + X_2 \geq 5$

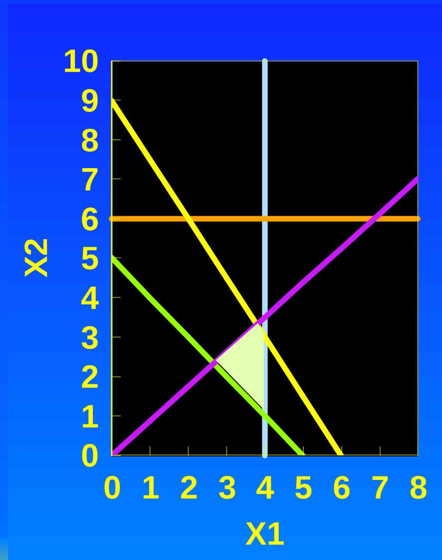
$3X_1 + 2X_2 \leq 18$

$-7X_1 + 8X_2 \leq 0$

$0 \leq X_1 \leq 4$

$0 \leq X_2 \leq 6$

- ► $X_1^* = 4$ $X_2^* = 1$



Solution Approach

1. Find a corner point
   – An "initial feasible solution"
2. Proceed to improved corner points
3. Stop when no further improvements are possible

Solution Methods

- Simplex
  - The textbook method
  - For step 2, select improved corners
    - *Always goes to best corner*
    - *Must seach all corners*
  - Inefficient for real problems
  - Not used in practice
- Practical methods - many exist - often proprietary
  - Step 2 takes many forms
  - Each best for different cases
  - Very great efficiency possible
  - A real art!

Typical Formulations: "Transportation" Problem

Objective = Minimize cost of moving a single commodity from sources "i" to uses "j"

$= \sum C_{ij} X_{ij}$

subject to: Amount shipped < Amount available

$\sum_j X_{ij} \leq S_i$

Amount delivered > Uses

$\sum_i X_{ij} \geq S_j$

Note:
Matrix of constraint coefficients are all 0's and 1's

- Particularly efficient solutions

Typical Formulations: "Blending" or "Diet" Problems

Objective = Minimize cost of materials

$= \sum C_i X_i$

subject to: Limits on availability

$X_i \leq$ Amounts given/available

Maxima or minima on impurities

trace elements, nutritional requirements

$\sum a_{ij} X_i \leq = \geq b_j$

Example:

Minimize cost of steel alloy when only so much scrap is available, subject to limitations on carbon content, tramp elements, etc.