Sensitivity Analysis

- Rationale
- Shadow Prices
  - Definition
  - Use
  - Sign
  - Range of Validity
- Opportunity Costs
  - Definition
  - Use

Rationale for Sensitivity Analysis

- Math problem is an approximation
  - optimum is an approximation
  - we need to check
- Constraints often artificial
  - Designer should question
    - *Should we have different specifications?*
- Situations always probabilistic
  - Prices change
  - Need to assess risk

Shadow Price Definitions

Recall from Constrained Optimization:

$$\text{Shadow price} = \left.\frac{\partial(\text{Objective})}{\partial(\text{Constraint})}\right|_{\text{optimum}}$$

Complementary Slackness:

Either (Slack variable) or (shadow price) = 0

Shadow Price Illustration

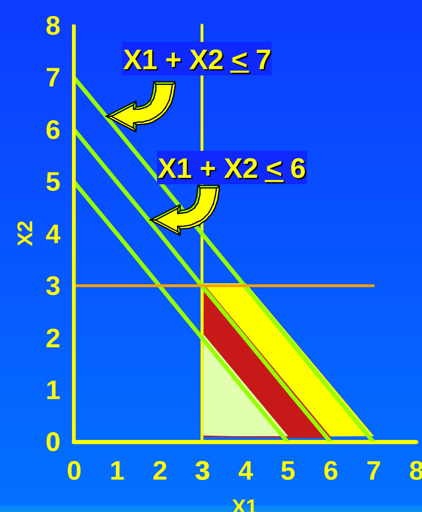
Max: $Y = X_1 + 4X_2$

s.t. $X_1 + X_2 \leq 5 = b_1$

$X_1 \geq 3 = b_2$

$X_2 \leq 3 = b_3$

$X_1 , X_2 \geq 0$



Notes:

a) $X_1^* = 3$; $X_2^* = 2$; $Y^* = 11$

b) when $\Delta b_1 = \pm 1$

$\Delta X_2^* = \pm 1$; $\Delta Y^* = \pm 4$; $SP_1 = 4$

c) $SP_3^* = 0$; $slack_3 = 1$

d) when $b_1 > 6$

$slack_3 = 0$; $SP_3 \neq 0$; $SP_1 = 1 \leq 4$

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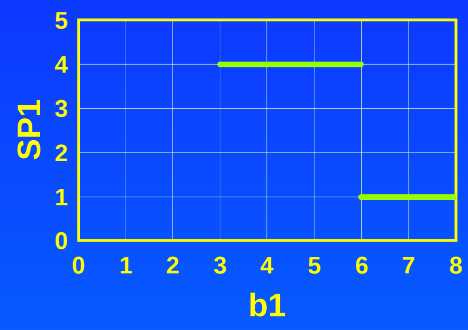
Use of Shadow Prices

- Proactive
  - Identify constraints with high S.P.
  - See if they can be changed for better solutions
    - *Example: New York water supply pressure < 60 psi at curb*
- Reactive
  - Respond to new opportunities
    - *Example: client changes specifications*
  - Respond to proposals for new constraints
    - *Example: trace chemicals*

Sign of Shadow Prices

- Note: "Obvious Rule" (+SP with +Δb) not correct
- Correct Reasoning:
  - What makes the optimum better?
    - Expansion of feasible region
    - "Relaxation of constraints"
  - What changes will increase the feasible region?
    - Increase upper bound
      $\Sigma_j a_{ij} X_j < b_i$
    - Decrease lower bound
      $\Sigma_k a_{kj} X_j > b_k$
  - i.e., "Raise the roof, lower the floor."

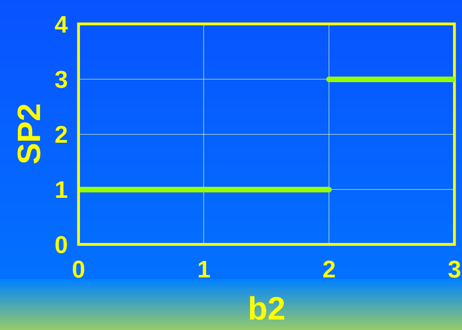
Shadow Prices As Constraints Change



increase an upper bound ("raise the roof")

decrease a lower bound ("lower the floor")

b_2: 3 --> 2
new X* = [2,3]
New Y* = 14
ΔY* = 3

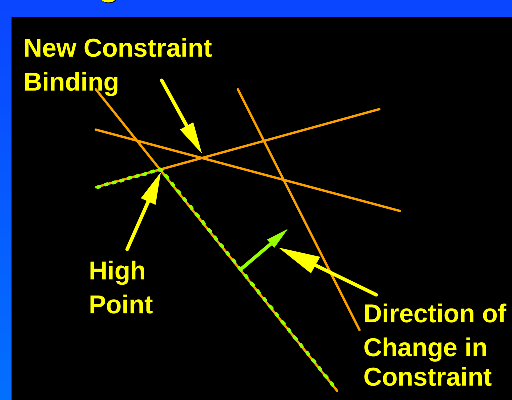


Range of Shadow Prices

- In Linear Programming, Shadow prices are constant
- Until a constraint changes enough so that a new constraint is binding
- Results given as:

  SP_K = constant
  for $r_L < b_K < r_U$

- Outside the range:
  - Shadow prices <u>decrease</u> as constraint is relaxed
  - Shadow prices <u>increase</u> as constraint is tightened



Opportunity Cost - Definition

- Objective Function = $\Sigma c_i X_i$
- Opportunity costs associated with c_i (coefficients of design variables)
- At optimum, somce decision variables = 0
  - These are non-optimal decision variables
- Opportunity cost is:
  - Degradation of optimum per unit of non-optimal variable introduced into design
  - A "cost" in that it is a <u>worsening</u> of optimum. Units may actually be almost anything; equal to whatever units are being optimized.

Meaning of Opportunity Costs

- Opportunity cost design trigger "price"
  - This is the value of the coefficient of the decision variable for which that variable should be in the design
- Suppose: Obj.Function = ... + $c_K X_K$ + ...
  and X_K not optimal with an
  opportunity cost = OC_K
- Then, as c_K changes for the better, (greater for maximization, lesser for minimization)

  OC_K lower
  $OC_K = 0$ at $c_K' = c_K - OC_K$
- c_K' is trigger price; defines the limit of best design

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Illustration of Opportunity Cost

- What happens when forced to use a non-optimal decision variable?
- Example: Min Cost = $2X_1 + 10X_2 + 20X_3$

  s.t. $X_1 + X_2 + X_3 \geq 3$

  $X_2 \geq 1$

  $X_1, X_2, X_3 \geq 0$
- $\underline{X}^* = (2, 1, 0)$; cost* = 14
- If forced to use X_3, new X* = (1,1,1); new cost* = 32
  (opportunity cost)$_3$ = $\Delta Z^*/1 = 18$

Meaning of Opportunity Cost

- If X_3 is used with no change in its unit cost (= c_3), the optimal cost would increase by 18
- If the cost of X3 were to fall by an amount equal to the opportunity cost ($c_3' = c_3 - OC_3 = 20-18 = 2$)